Reducibility candidates modulo isomorphisms

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Loria
Laboratoire lorrain de recherche en informatique et ses applications

Inria

Outline

Proposal

we have **System F/$_\sim$** we want $\mathcal{SN}$ we need $RED/_\sim$

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we have **System F/$_\sim$** we want $\mathcal{SN}$ we need $RED/_\sim$

Outline

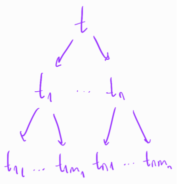
- Termination
- Reducibility
  - STLC
  - System F
  - System F modulo isomorphisms

What and why

Intuition

$$(\lambda x.\, t)\; s$$

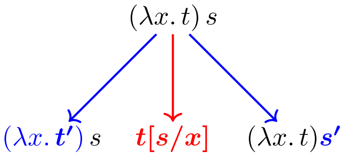
$$(\lambda x.\, \boldsymbol{t'})\; s \qquad \boldsymbol{t[s/x]} \qquad (\lambda x.\, t)\,\boldsymbol{s'}$$

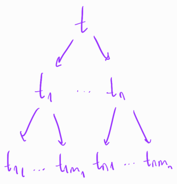


$\mathcal{SN}$ = all branches finite

What and why

Intuition





$\mathcal{SN}$ = all branches finite

Why?

- safety core language (without fix)
- freedom at implementation
- (my take) should–have unless intended

How (not by induction on terms)

Induction does not work

$$\overbrace{(\lambda x.\, t)\; s}^{\text{new untested redex}}$$

$$\downarrow$$

$$\underbrace{\boldsymbol{t[s/x]}}_{>(\lambda x.t)s}$$

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$\mathcal{SN}$ of subterms is **not enough**

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We need more: also remain $\mathcal{SN}$ when applied

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$$(\lambda x.\, t)\; s\; u$$

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Induction does not work

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$$\underbrace{\boldsymbol{t[s/x]}}_{\mathcal{SN} \text{ by IH}}\; u$$

Remaining $\mathcal{SN}$ is **not enough**

We need more: recursively remain $\mathcal{SN}$ when applied

Reducibility for STLC — Fetching stage

Intuition

- We need terms to behave well under all possible uses
- We need to know all the possible uses

$$t: \quad A_1 \quad \to \quad \cdots \quad \to \quad A_n \quad \to \quad \tau$$
$$\downarrow$$

Reducibility for STLC Fetching stage

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The RED–set

$$\begin{array}{rcl} [\![\tau]\!] & = & \mathcal{SN}_\tau \\ [\![A \to B]\!] & = & \{\, t \in A \to B \mid \forall s \in [\![A]\!].\ ts \in [\![B]\!] \,\} \end{array}$$

Reducibility for STLC — Testing stage

Intuition

test $\lambda x.t$ for $[\![A \to B]\!]$

$$(\lambda x.t)s$$

$$(\lambda x.\boldsymbol{t'})s \qquad \boldsymbol{t}[\boldsymbol{s}/\boldsymbol{x}] \qquad (\lambda x.\,t)\boldsymbol{s'}$$

CR3: neutrality + induction on B: all one–step reducts RED implies RED

Reducibility for System F — Problem

Unlike $\lambda^{\to}$, not so easy to find the RED–set

Type application substitutes both the term *and the type*

$$\Lambda X.t : \forall X.A \qquad (\Lambda X.t)B : A[B/X]$$

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$$[\![\forall X.A]\!] \;=\; \{\, t \in \forall X.A \mid \forall B \in \mathcal{K}.\ tB \in [\![A[B/X]]\!] \,\}$$

Example Let $I_\forall = \forall X.X \to X$

$$t : \quad \forall X \quad X \quad \to \quad X$$

$$t$$

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$$t \qquad X$$

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Example Let $I_\forall = \forall X.X \rightarrow X$

$$t : \forall X \quad X \rightarrow X$$

$$\Uparrow$$

$$t \quad X \quad x^X \in \mathcal{SN}_X$$

$$t : \forall X \quad X \rightarrow X$$

$$t \quad I_\forall$$

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Parametric RED–set

- Avoid substitution: stop at X
- Parameterize: save $I_\forall$ into a mapping $\rho : \mathsf{TVar} \to$ RED-set

Reducibility for System F

Fetching stage

But...

what to put into ρ for B?

B is a type   we can't put $[\![B]\!]$ what RED–set?

Reducibility for System F — Fetching stage

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Candidates

1. Abstractly describe RED–set by properties

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any set satisfying CR1, CR2 and CR3

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   any set satisfying CR1, CR2 and CR3
3. Parameterize $[\![\cdot]\!]$ by a map $\rho : \mathsf{TVar} \to \mathcal{R}$

$$[\![X]\!]_\rho \quad = \quad \rho(X)$$

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3. Parameterize $[\![\cdot]\!]$ by a map $\rho : \mathsf{TVar} \to \mathcal{R}$

$$[\![X]\!]_\rho \quad = \quad \rho(X)$$

4. Make $\forall$–step range over all $\mathcal{R}$ for any type B

$$[\![\forall X.A]\!]_\rho \quad = \quad \{\, t \in \forall X.A \mid \forall B \in \mathcal{K}, \mathcal{C}_B \in \mathcal{R}_B.\ tB \in [\![A]\!]_{[\rho \cdot X \mapsto \mathcal{C}_B]} \,\}$$

Reducibility for System F — Testing stage

when testing $\Lambda X.t : \forall X.A$

$$(\Lambda X.t)B$$

$$\swarrow \qquad \searrow$$

$$(\Lambda X.t)B \qquad \boldsymbol{t[B/X]}$$

tests from $[\![A]\!]_{\rho\cdot[\mathcal{C}_B/X]}$ are left to the CR3 induction

Simply typed λ–calculus modulo isomorphisms

Equivalence on types

$$A \to (B \times C) \quad \sim \quad (A \to B) \times (A \to C)$$

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Typing rules

$$\frac{\Gamma \vdash t : A \qquad A \sim B}{\Gamma \vdash t : B}$$

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Equivalence on terms

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Reduction

$$\to \;::=\; \rightleftarrows^* \circ \hookrightarrow$$

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Reduction

$$\to \; ::= \; \rightleftarrows^* \circ \hookrightarrow$$

$$\langle \lambda x.x, \lambda x.\lambda y.x\rangle s \;\rightleftarrows\; (\lambda x.\langle x, \lambda y.x\rangle)s \;\hookrightarrow\; \langle s, \lambda y.s\rangle$$

(Simple) Reducibility modulo isomorphisms — Problems

Problems

Problem 1: fetching (later) Problem 2: testing

(Simple) Reducibility modulo isomorphisms — Problems

Problems

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Problem 2: testing

Problem 2.a: Lack of neutrality

Problem 2.b: Knowing all the u_i

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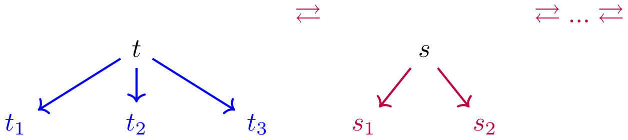
Testing



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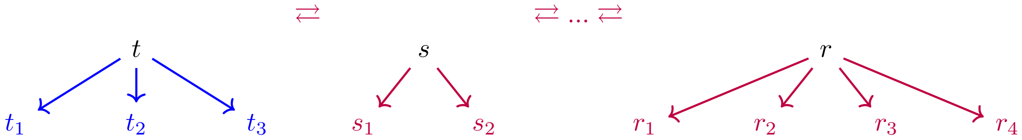

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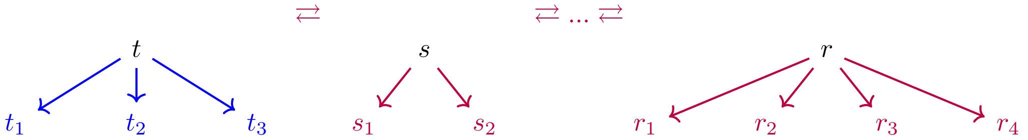
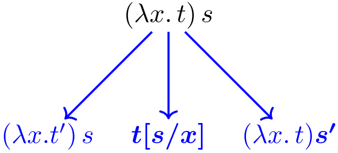
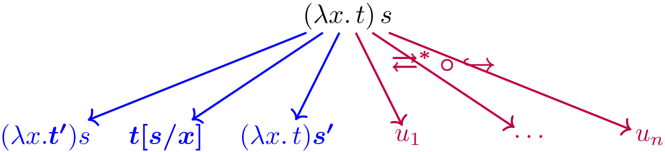


STLC



STLC/$\sim$



(Simple) Reducibility modulo isomorphisms — Solution 2.a

Solving lack of neutrality

- Neutrality is used to test one eliminator at a time
- Constructors commutation breaks neutrality
- "Local" testing does not work

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$$[\![A_1 \to \ldots \to A_n \to \tau]\!]$$

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Fetching all at once

$$[\![A_1 \to \ldots \to A_n \to \tau]\!]$$

Testing all possible eliminators $\vec{u} = (u_1, \ldots, u_n)$ from $[\![A_1]\!] \times \ldots \times [\![A_n]\!]$

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- Constructors commutation breaks neutrality
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Fetching all at once

$$[\![A_1 \to \ldots \to A_n \to \tau]\!]$$

Testing all possible eliminators $\vec{u} = (u_1, \ldots, u_n)$ from $[\![A_1]\!] \times \ldots \times [\![A_n]\!]$

$$(\lambda x.t)s\vec{u}$$

$(\lambda x.\boldsymbol{t'})s\vec{u}$ $\quad$ $\boldsymbol{t[s/x]}\vec{u}$ $\quad$ $(\lambda x.t)\boldsymbol{s'}\vec{u}$ $\quad$ $\boldsymbol{(\lambda x.t)}s\vec{u}'$

(Simple) Reducibility modulo isomorphisms — Solution 2.a

Solving lack of neutrality

- Neutrality is used to test one eliminator at a time
- Constructors commutation breaks neutrality
- "Local" testing does not work

Fetching all at once

$$[\![A_1 \to \ldots \to A_n \to \tau]\!]$$

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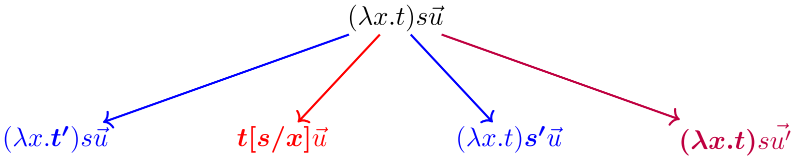

Remark the proof obligation is now $\mathcal{SN}$ instead of RED

System F (polymorphic λ–calculus) modulo isomorphisms

Equivalence on types

$$A \to (B \times C) \sim (A \to B) \times (A \to C)$$
$$\forall X.(A \times B) \sim (\forall X.A) \times (\forall X.B)$$

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$$\frac{\Gamma \vdash t : A \qquad A \sim B}{\Gamma \vdash t : B}$$

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Equivalence on terms

$$\lambda x.\langle t, s\rangle \quad \leftrightarrows \quad \langle \lambda x.t, \lambda x.s\rangle$$
$$\langle t, s\rangle u \quad \leftrightarrows \quad \langle tu, su\rangle$$
$$\Lambda X.\langle t, s\rangle \quad \leftrightarrows \quad \langle \Lambda X.t, \Lambda X.s\rangle$$
$$\langle t, s\rangle A \quad \leftrightarrows \quad \langle tA, sA\rangle$$

System F (polymorphic λ–calculus) modulo isomorphisms

Equivalence on types

$$
\begin{array}{rcl}
A \to (B \times C) & \sim & (A \to B) \times (A \to C) \\
\forall X.(A \times B) & \sim & (\forall X.A) \times (\forall X.B)
\end{array}
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\langle t, s\rangle A & \leftrightarrows & \langle tA, sA\rangle
\end{array}
$$

Reduction

$$
\to \; ::= \; \rightleftarrows^* \circ \hookrightarrow
$$

Reducibility for System F/∼ — Problem 2.a again

Testing/∼

we agreed on testing "globally"

$$(\Lambda X.t)B\vec{u}$$

$$(\Lambda X.\boldsymbol{t'})B\vec{u} \qquad \boldsymbol{t[B/X]}\vec{u} \qquad (\Lambda X.t)B\vec{\boldsymbol{u'}}$$

Reducibility for System F/$\sim$ — Problem 2.a again

Testing/$\sim$ we agreed on testing "globally"

$$(\Lambda X.t)B\vec{u}$$

$$\swarrow \qquad \downarrow \qquad \searrow$$

$$(\Lambda X.\boldsymbol{t'})B\vec{u} \qquad \boldsymbol{t[B/X]}\vec{u} \qquad (\Lambda X.t)B\vec{\boldsymbol{u'}}$$

But...

which are the possible $\vec{u}$ of $[\![X]\!]_\rho$?

- $[\![X]\!]_\rho = \rho(X)$ is any candidate
- we only know CR1, CR2 and CR3
- we don't have enough information to range over $\vec{u}$

Alternative RED for System F: Parigot candidates

Giving structure to candidates

Alternative RED for System F: Parigot candidates

Giving structure to candidates

Family of sets of candidates inductively!

$$\frac{}{\mathcal{SN}_A \in \mathcal{R}_A} \qquad \frac{U \in \mathcal{R}_A \quad V \in \mathcal{R}_B}{U \tilde{\rightarrow} V \in \mathcal{R}_{A \to B}}$$

$$\frac{X \subseteq \mathcal{R}_A}{\bigcap X \in \mathcal{R}_A} \qquad \frac{(U_B \in \mathcal{R}_{A[B/X]})_{B \in \mathcal{K}}}{\tilde{\forall} B.U_B \in \mathcal{R}_{\forall X.A}}$$

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Example

$$\frac{\overline{\mathcal{SN}_{A_1} \in \mathcal{R}_{A_1}} \qquad \begin{array}{c}\dfrac{\overline{\mathcal{SN}_{A_n} \in \mathcal{R}_{A_n}} \qquad \overline{\mathcal{SN}_X \in \mathcal{R}_X}}{\mathcal{SN}_{A_n} \tilde{\rightarrow} \mathcal{SN}_X \in \mathcal{R}_{A_n \rightarrow X}} \\ \vdots \end{array}}{\mathcal{SN}_{A_1} \tilde{\rightarrow} \dots \tilde{\rightarrow} \mathcal{SN}_{A_n} \tilde{\rightarrow} \mathcal{SN}_X \in \mathcal{R}_{A_1 \rightarrow \dots \rightarrow A_n \rightarrow X}}$$

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Example

$$\frac{\frac{}{\mathcal{SN}_{A_1} \in \mathcal{R}_{A_1}} \qquad \frac{\vdots \qquad \frac{\mathcal{SN}_{A_n} \in \mathcal{R}_{A_n} \qquad \mathcal{SN}_X \in \mathcal{R}_X}{\mathcal{SN}_{A_n} \tilde{\to} \mathcal{SN}_X \in \mathcal{R}_{A_n \to X}}}{}}{\mathcal{SN}_{A_1} \tilde{\to} \dots \tilde{\to} \mathcal{SN}_{A_n} \tilde{\to} \mathcal{SN}_X \in \mathcal{R}_{A_1\to\dots\to A_n\to X}}$$

Fetching

- the RED–set

$$[\![A \to B]\!]_\rho = \frac{[\![A]\!]_\rho \in \mathcal{R}_A \qquad [\![B]\!]_\rho \in \mathcal{R}_B}{[\![A]\!]_\rho \tilde{\to} [\![B]\!]_\rho \in \mathcal{R}_{A\to B}}$$

- the eliminators $\vec{u}$

$$[\![\overline{\mathcal{SN}_X \in \mathcal{R}_X}]\!]_\rho^\perp = \varepsilon$$

Alternative RED for System F: Parigot candidates

Giving structure to candidates

Family of sets of candidates inductively!

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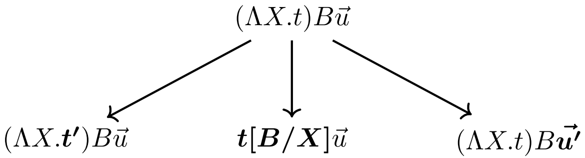
$$\vdots$$

$$[\![\overline{\mathcal{SN}_A \tilde{\to} \mathcal{SN}_B \in \mathcal{R}_{A \to B}}]\!]^\perp_\rho = (u)_{u \in \mathcal{SN}_A}$$

Reducibility for System F/$\sim$ — Problem 2.a solved

Testing/$\sim$



Now

which are the possible $\vec{u}$ of $[\![X]\!]_\rho$?

- $[\![X]\!]_\rho = \rho(X)$ is any candidate
- we only know ~~CR1, CR2 and CR3~~ **have structure for $\rho(X)$**
- we ~~don't~~ **do!** have enough information
- **range over the eliminators $\vec{u}$ in $\rho(X)$**

RED for System F/$\sim$ Fetching/$\sim$

Problem 1

Against which terms should $\langle \lambda x^A.t, \lambda x^A.s \rangle$ be tested?

RED for System F/$\sim$ — Fetching/$\sim$

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But modulo isomorphisms

- types are part of an equivalence class
- restrictions comes from *all* the class

$$\langle \lambda x^A.t, \lambda x^A.s \rangle$$

$$(A \to B) \times (A \to C) \qquad A \to (B \times C)$$

RED for System F/$\sim$ — Fetching/$\sim$

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Recall that

- $[\![\cdot]\!]_.$ follows types fetching restrictions
- by induction $m(A)$
- $m(A)$ is a stable measure on types

RED for System F/$\sim$ — Fetching/$\sim$

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Recall that

- $[\![\cdot]\!].$ follows types fetching restrictions
- by induction $m(A)$
- $m(A)$ is a stable measure on types

$$\begin{aligned}
[\![A]\!]_\rho \quad = \quad & \textstyle\bigcap_{A_1\times A_2\sim A}\{\,[\![A_1]\!]_\rho \,\tilde{\times}\, [\![A_2]\!]_\rho\,\}\\
\cap \quad & \textstyle\bigcap_{A_1\to A_2\sim A}\{\,[\![A_1]\!]_\rho \,\tilde{\to}\, [\![A_2]\!]_\rho\,\}\\
\cap \quad & \textstyle\bigcap_{\forall X.A'\sim A}\{\,\tilde{\forall} B. \cap\{\,[\![A']\!]_{\rho\cdot[\mathcal{C}_B/X]}\,\}\,\}
\end{aligned}$$

RED for System F/$\sim$ — Parigot candidates/$\sim$

Changes in the family $\mathcal{R}_*$ due to isomorphisms

RED for System F/$\sim$ Parigot candidates/$\sim$

Changes in the family $\mathcal{R}_*$ due to isomorphisms

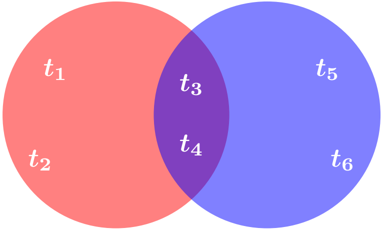
Family of sets of candidates inductively!

$$\frac{}{\mathcal{SN}_A \in \mathcal{R}_A} \qquad \frac{U \in \mathcal{R}_A \quad V \in \mathcal{R}_B}{U \tilde{\to} V \in \mathcal{R}_{A\to B}}$$

$$\frac{X \subseteq \mathcal{R}_A}{\bigcap X \in \mathcal{R}_A} \qquad \frac{(U_B \in \mathcal{R}_{A[B/X]})_{B\in\mathcal{K}}}{\tilde{\forall} B.U_B \in \mathcal{R}_{\forall X.A}}$$

$$\frac{F \in \mathcal{R}_A \quad G \in \mathcal{R}_B \quad A \equiv B}{F \cap G \in \mathcal{R}_A}$$

$A \to (B \times C)$ $\qquad$ $(A \to B) \times (A \to C)$



RED for System F/$\sim$ Testing/$\sim$

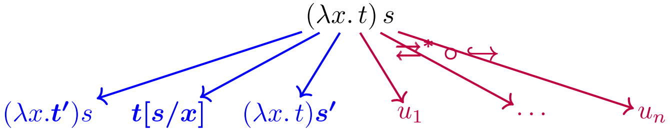
Problem 2.b

What are the one–step reducts of a term?

RED for System F/∼ — Testing/∼

Problem 2.b

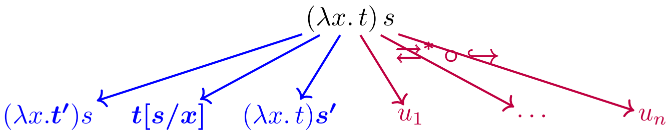
What are the one–step reducts of a term?



RED for System F/$\sim$ — Testing/$\sim$

Problem 2.b

What are the one–step reducts of a term?



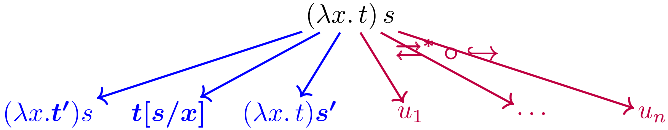
Solution

1. characterize classes of terms

RED for System F/$\sim$ — Testing/$\sim$

Problem 2.b

What are the one–step reducts of a term?



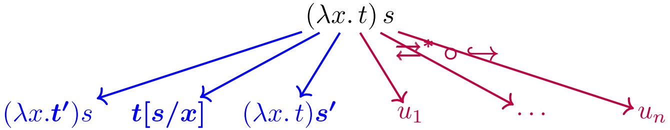
Solution

1. characterize classes of terms
2. look at the one–step redex of each shape

RED for System F/∼ — Testing/∼

Problem 2.b

What are the one–step reducts of a term?

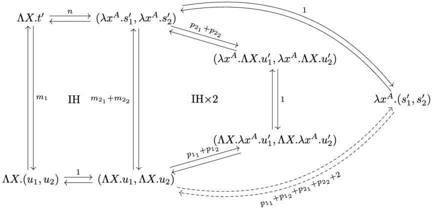


Solution

1. characterize classes of terms
2. look at the one–step redex of each shape

Lemma 3.2 (The class of type abstractions). *If $\Lambda X.t' \rightleftarrows^n s$, then s is equal to:*

1. *$\Lambda X.s'$ with $t' \rightleftarrows^m s'$ and $m \leq n$*
2. *$\lambda x^A.s'$ with $t' \rightleftarrows^{m_1} \lambda x^A.r$, $s' \rightleftarrows^{m_2} \Lambda X.r$, $m_1+1+m_2 \leq n$, and $X \notin FV(A)$*
3. *$\langle s'_1, s'_2\rangle$ with $t' \rightleftarrows^{m_1} \langle r_1, r_2\rangle$, $s'_i \rightleftarrows^{m_{2_i}} \Lambda X.r_i$, and $m_1 + 1 + m_{2_1} + m_{2_2} \leq n$*
4. *$\pi_{\forall X.A}s'$ with $t' \rightleftarrows^{m_1} \pi_A r$, $s' \rightleftarrows^{m_2} \Lambda X.r$, and $m_1 + 1 + m_2 \leq n$*



Conclusions

- We need to prove $\mathcal{SN}$ in a System F modulo isomorphisms
- The calculus has no neutrality
- Parigot's approach to reducibility does not use neutrality
- We are adapting Parigot's technique to modulo isomorphisms by
  1. relating candidates of isomorphic types
  2. fetching restrictions from all the equivalents
  3. characterizing the one–step reducts of all term–classes