New decreasing measures for typed λ-calculi

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CONICET

UBA
Universidad de Buenos Aires
Argentina virtus robur et studium

Universidad Nacional de Quilmes

ICC
Instituto de Ciencias de la Computación

Self-introduction

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  λ-calculus modulo type isomorphism

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- Visiting Researcher @ IRIF – September 2026 - March 2027
  - $\dfrac{\text{Useful CbV strategy variant}}{\text{VSC}} \quad \dfrac{\text{tight typing system}}{\text{LVSC} \iff \text{U(RL)VSC}}$ with Delia Kesner & Pablo Barenbaum
  - A typing system for the Positive λ-calculus with Delia Kesner & Giulio Manzonetto

(Non-academic) Self-introduction

- Like talking about politics

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- Like talking about politics
- Worried about... the world

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- I am Argentinian $\Longrightarrow$ drink mate

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- I don't eat candy

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Termination in the λ-calculus

$$t, s, \ldots \quad ::= \quad x \quad | \quad \lambda x.t \quad | \quad t\, s$$

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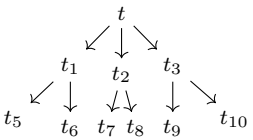
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Reduction tree



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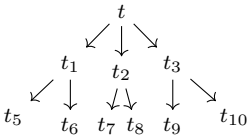
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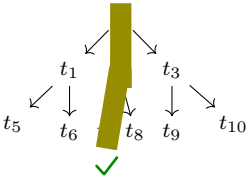
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Reduction tree



Weak normalization WN



Termination in the λ-calculus

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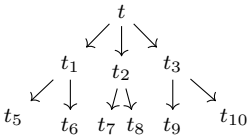
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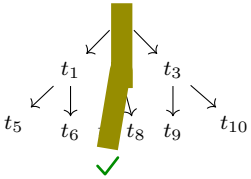
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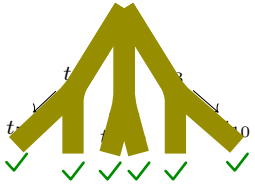
Reduction tree



Weak normalization WN



Strong normalization (SN)



Termination in the λ-calculus

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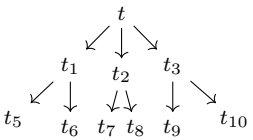
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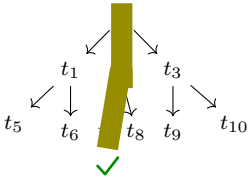
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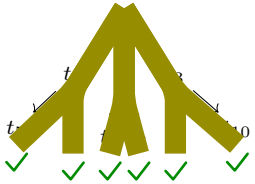
Reduction tree



Weak normalization WN



Strong normalization (SN)

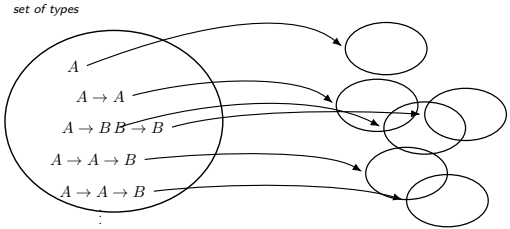


Proving SN usual

- safest system / strongest result
- hard (and beautiful!) problem

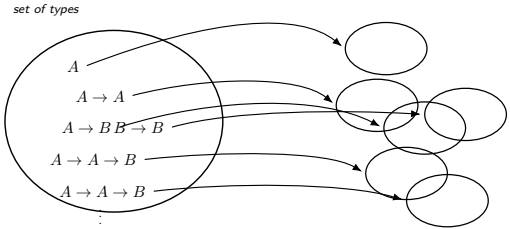
Reducibility (candidates) [Tait'67, Girard'71/72]

- Interpreting types as sets of terms
- Capturing "well-behaviour" w.r.t. termination



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abstraction

concise — **lack intuition**

adaptable — **why it works?**

The complexity of reducibility

Abstraction

de Vrijer'87

The *loose valuation* only gives an upper bound for the height. But the proof of strong normalization that corresponds to it—here called the "quick proof"—is simple and transparent. Presumably more transparent than the ones involving a computability predicate. As a matter of fact the quick proof can be viewed as providing some extra insight into the nature of the computability proofs. The computability predicate should then be analyzed as abstracting from all the information—still explicit in the labeled functionals—that is not strictly needed in an induction on terms which successfully establishes strong normalization.

Insight on *why*

Gallier'94

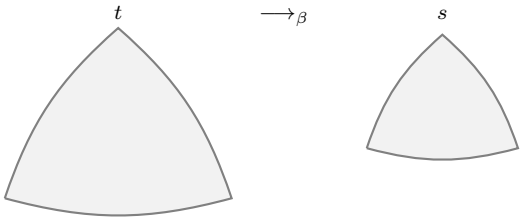
This paper provides some answers to the above questions. But before explaining our results, we would like to explain our motivations and our point of view a little more. Reducibility proofs are seductive and thrilling, but also elusive. Following these proofs step-by-step, we see that they "work" (when they are not wrong!), but I claim that most of us would still admit that they are not sure *why* these proofs work! The situation is somewhat comparable to driving a Ferrari (I suppose): the feeling of power is tremendous, but what exactly is under the hood? What kind of carburator, what kind of valve mechanism, gives such power and flexibility?

Decreasing measures

Decreasing measure

$\#(\cdot) : \lambda\text{-terms} \to \text{well-founded order}$ $\qquad t \to_\beta s \implies \#(t) > \#(s)$

Intuitively

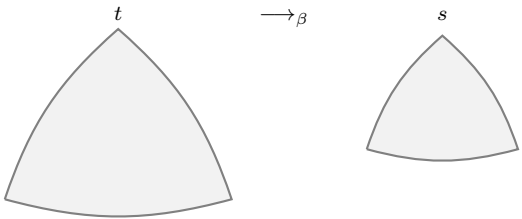


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Why?

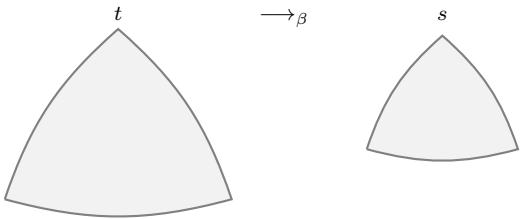
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Why?

- Information
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Existing measures

- Gandy'80
- de Vrijer'87

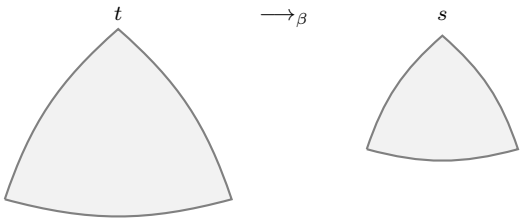
translation to λ-I
+
interpretation on HSMF
(hereditarily strictly monotonic functions)

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Why?

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More syntactic (staying within LC)?

Works

Four decreasing measures for typed λ-calculi

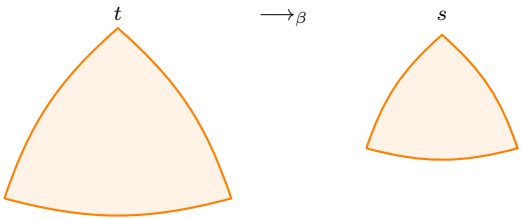
- $\mathcal{W}$: simple measure for STLC [Barenbaum, S., FSCD'23]
- $\mathcal{W}_\cap$: extension to IIT [Barenbaum, Ronchi Della Rocca, & S., MFPS'25]
- $\hat{\mathcal{W}}$: refinement to exactness STLC [To be submitted]
- $\mathcal{T}^{\mathbb{m}}$: complex measure for STLC [Barenbaum & S., FSCD'23]
- $\hat{\mathcal{W}}_\cap$: refinement to exactness for $\Lambda_\cap$ (WIP)

Gödel's koan (open question)

[TLCA #26, RTA #19]

Besides *merely* finding new ways of proving termination... we have Gödel's koan

To find an "easy" mapping from simply typed λ–calculus to ordinals



Koan?

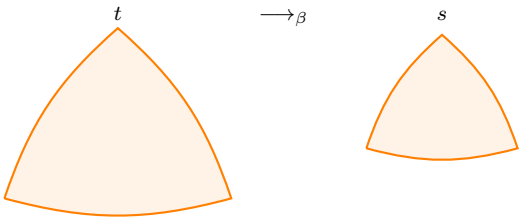
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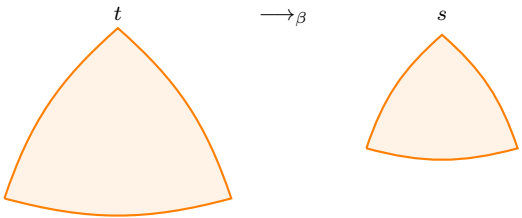
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- define
- compute
- prove
- understand

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Complexity dimensions

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The koan in these works

- We did not solve it
- Easy at some dimensions, but not all
- We contributed new ideas
- Remains as a long-term motivation

$\mathcal{W}$: A simple decreasing measure for STLC

[Barenbaum & Sottile, FSCD'23]

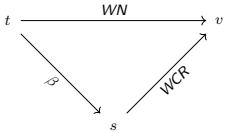
Concretized one from known ideas

DECrease from INCrease

Nederpelt-Klop's lemma $INC \wedge WCR \wedge WN \implies SN \wedge CR$

DECrease from INCrease

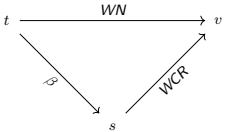
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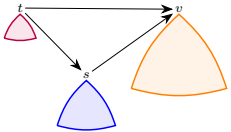
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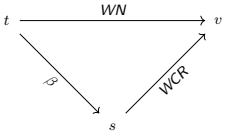


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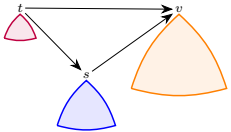


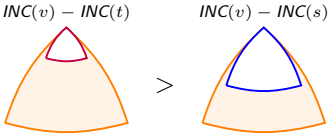
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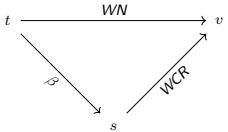
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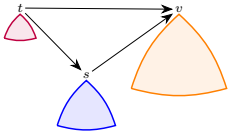
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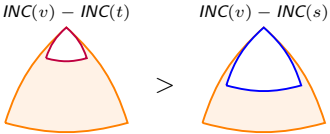


Recipe for DM!

$INC \wedge WCR \wedge WN \implies SN \wedge CR$



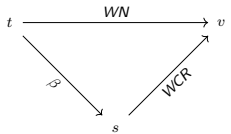
But... not for STLC: not INC



What about a "superset" of STLC?

DECrease from INCrease

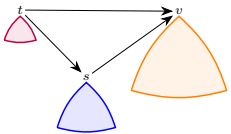
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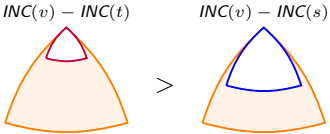
Recipe for DM!

The memory calculus $\lambda^{\mathbf{m}}$

$INC \wedge WCR \wedge WN \implies SN \wedge CR$



But... not for STLC: not INC



What about a "superset" of STLC?

Avoid erasure

$(\lambda x.y)t \to_{\mathbf{m}} y\langle t \rangle$

- Known idea (Church, Nederpelt, Klop)
- INC (more memories)
- WCR (by usual means)

Redex degrees and Turing's observation

Degree of a redex

Height of type of abstraction

$$\underbrace{(\lambda x.t)}_{A \to B}\, s$$

Redex degrees and Turing's observation

Degree of a redex

Height of type of abstraction

$$\underbrace{(\lambda x.t)}_{A \to B}\, s$$

Turing's observation

Contracting redex $R := M \to_\beta N$ creates redex S

$\Longrightarrow$

degree of R is **greater** than degree of S

Redex degrees and Turing's observation

Degree of a redex

Height of type of abstraction

$$\underbrace{(\lambda x.t)}_{A\to B}\ s$$

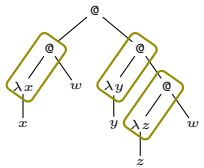
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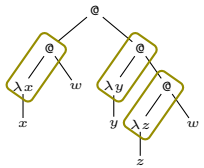
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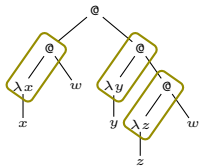
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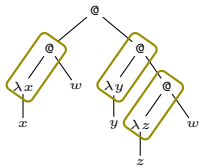
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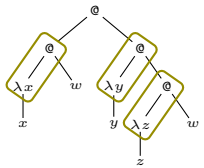
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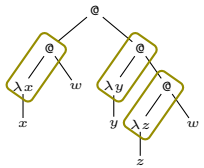
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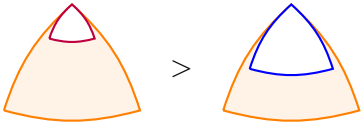
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Measure $\mathcal{W}$

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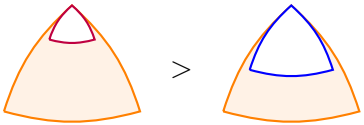
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$\lambda^{\mathbf{m}}$ measure, not STLC measure

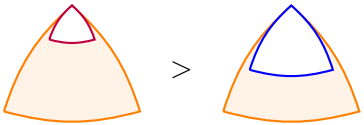
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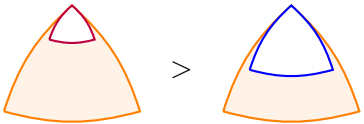
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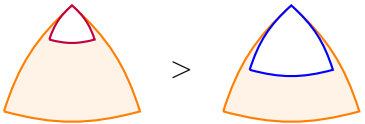
Main theorem

$$M \rightarrow_\beta N \implies \mathcal{W}(M) > \mathcal{W}(N)$$

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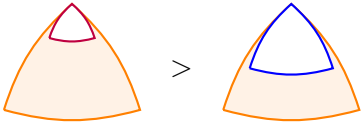
Erases a memory $\qquad t\langle s\rangle \rhd t$

- Relate $\lambda^{\mathbf{m}}$ and STLC terms
- Commutes with $\to_{\mathbf{m}}$

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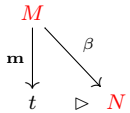
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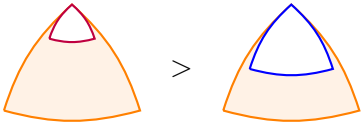
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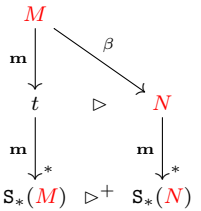
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Quick conclusion

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Turing's observation

Nederpelt-Klop's lemma ⇓ ⇓

Klop's non-erasure ⇓

new strong DM

Gandy's strong DMs

$\mathcal{W}_\cap$: A simple measure for $\Lambda_\cap$

[Barenbaum, Ronchi Della Rocca & Sottile, MFPS'25]

Adapted $\mathcal{W}$ to $\Lambda_\cap$

Introduction

Extra motivation

Existing measures weak or STLC-translation based $\Longrightarrow$ no native strong measure

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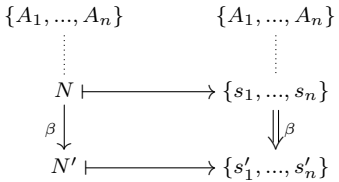
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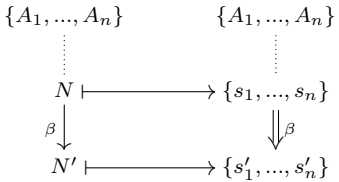
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S_d: $\neq$ type, $\neq$ degree

$\{s_1, \quad ..., \quad s_n\}$

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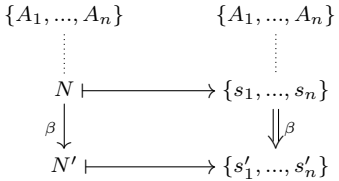
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$\Lambda_\cap^{\mathsf{Ch}}$: Idempotent intersection types à la Church

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Linear reduction Inspired by Kfoury's

$$\{s_1, ..., t_n\} \underset{\mathsf{Ch}}{\longrightarrow} ... \longrightarrow \{s_1, ..., s_n\}$$

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$$\{A_1, ..., A_n\} (A_i \neq A_j \text{ if } i \neq j)$$

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$$\downarrow_{\mathsf{Ch}}$$

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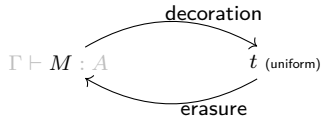
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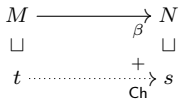
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Theorems relating $\Lambda^{\mathsf{Cu}}_\cap$ and $\Lambda^{\mathsf{Ch}}_\cap$

Correspondence



Simulation



Bijection between set-term $\vec{s}$ and set-type $\vec{A}$

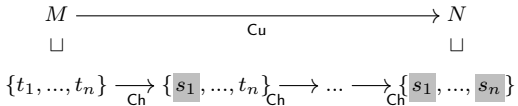
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Measure

- Extend $\Lambda_\cap^{\mathsf{Ch}}$ to memory $\Lambda_\cap$

Measure

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- Replicate $\mathcal{W}$: obtain $\mathcal{W}_\cap^{\mathsf{Ch}}$

Measure

- Extend $\Lambda_\cap^{\mathsf{Ch}}$ to memory $\Lambda_\cap$
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- One more step: lifting to $\Lambda_\cap^{\mathsf{Cu}}$

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Curry-style measure

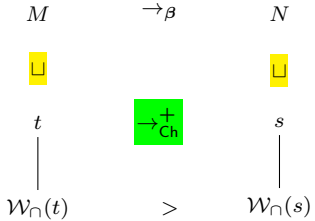
measure of the Church-correspondent

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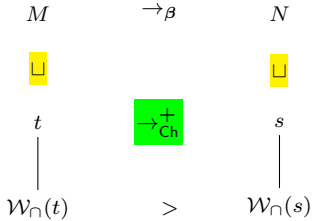


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Main theorem

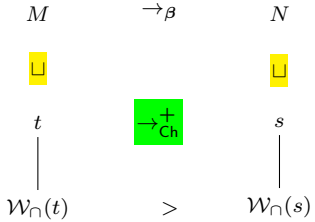
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$$M \to_\beta N \implies \mathcal{W}_\cap(M) > \mathcal{W}_\cap(N)$$

Completing the picture: typable iff SN

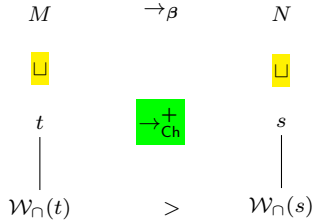
$$\Gamma \vdash M : A \iff M \in SN$$

Measure

- Extend $\Lambda_\cap^{\mathsf{Ch}}$ to memory $\Lambda_\cap$
- Replicate $\mathcal{W}$: obtain $\mathcal{W}_\cap^{\mathsf{Ch}}$
- One more step: lifting to $\Lambda_\cap^{\mathsf{Cu}}$

Curry-style measure

measure of the Church-correspondent



Main theorem

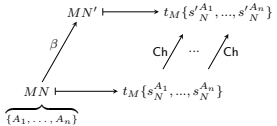
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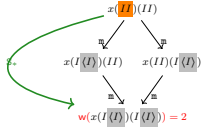
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Quick conclusion

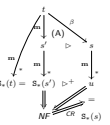
Church-style IIT



First native strong DM



Simplest codomain: $\mathbb{N}$



$\hat{\mathcal{W}}$ & $\hat{\mathcal{W}}_\cap$: Refinement to exactness of $\mathcal{W}$ & $\mathcal{W}_\cap$

ongoing work with Barenbaum & Ronchi Della Rocca

Obtaining the tightest possible value

$\hat{\mathcal{W}}$: exact refinement for $\mathcal{W}$

$\mathbb{N}$ measures $\implies$ upper bound for $|M|$ **Is the upper bound exact?**

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Fixing overmeasurement

Problem $\rightarrow_{\mathbf{m}}$ always remembers

$$(\lambda x.x)s \xrightarrow[\mathbf{m}]{} s\langle s\rangle$$
$$\beta \downarrow \quad \nearrow\!\!\nearrow$$
$$s$$

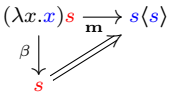
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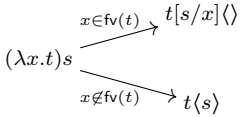
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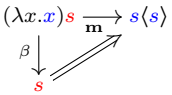
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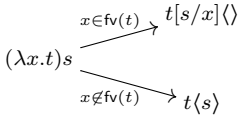
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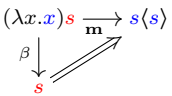
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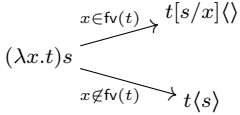
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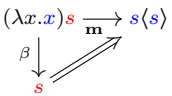
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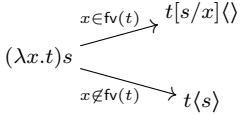
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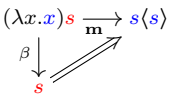
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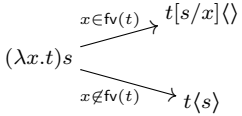
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Extra problems Curry vs. Church gap

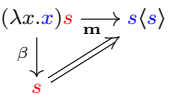
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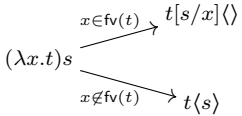
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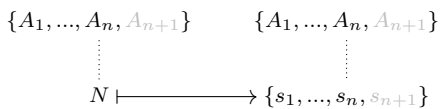
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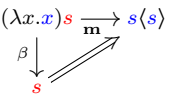
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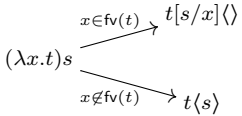
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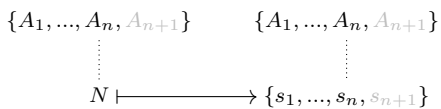
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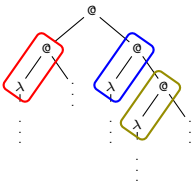
$\mathcal{T}^{\mathbb{m}}$: A complex decreasing measure for STLC

[Barenbaum & Sottile, FSCD'23]

Extending Turing's measure

Measure $\mathcal{T}^{\mathbb{m}}$: Extending Turing's result

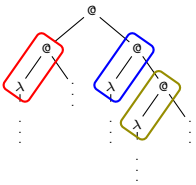
Turing's WN measure



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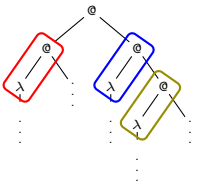
Adapt it to prove SN

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redex contraction
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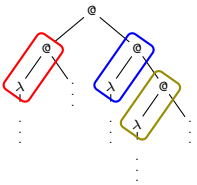
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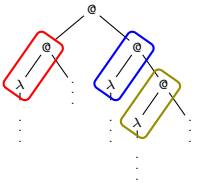
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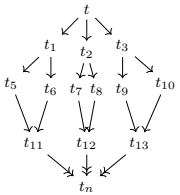
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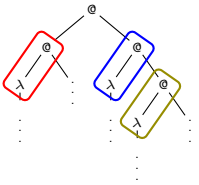
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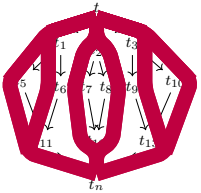
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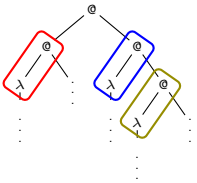
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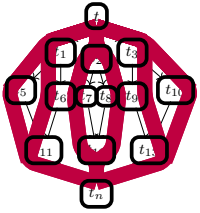
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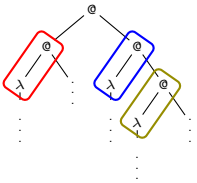
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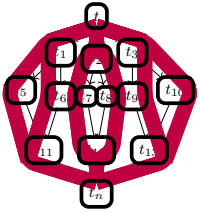
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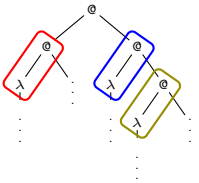
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$$\underbrace{[\mathcal{T}_{d-1}(t), \mathcal{T}_{d-1}(t_1), ..., \mathcal{T}_{d-1}(t_n)]}_{[\ \mathcal{T}^{\mathfrak{m}}_{d-1}(t')\ ||\ \rho{:}t \xrightarrow{d}_{\mathbf{m}}{}^{*} t'\ ]}$$

Plan

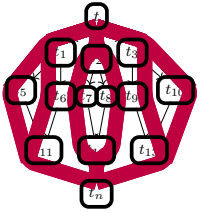
1. Indexed family of measures
2. Collect more information

$$\mathcal{T}_3(t) = [(3,?),(2,?),(1,?)]$$
$$\mathcal{T}_2(t) = [(2,?),(1,?)]$$
$$\mathcal{T}_1(t) = [(1,?)]$$

Lower measure: *e.g.*, $\mathcal{T}^{\mathfrak{m}}_2(t)$ when defining $\mathcal{T}^{\mathfrak{m}}_3(t)$

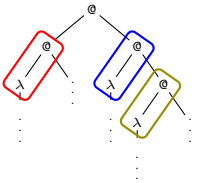
Consider:

1. d-red: $t \xrightarrow{d} s$ (Conf. & SN)
2. d-devs: $t \xrightarrow{d}{}^{*} s$



Measure $\mathcal{T}^{\mathfrak{m}}$: Extending Turing's result

Turing's WN measure



$$\mathcal{T}(t) \quad = \quad [\ 3\ ,\ 2\ ,\ 1\ ]$$

Central idea

Adapt it to prove SN

Problem

redex contraction
can copy greater redexes
$\therefore$ growing MS

$$\underbrace{[\mathcal{T}_{d-1}(t), \mathcal{T}_{d-1}(t_1), ..., \mathcal{T}_{d-1}(t_n)]}_{[\ \mathcal{T}^{\mathfrak{m}}_{d-1}(t')\ ||\ \rho : t \xrightarrow{d}{}^{*}_{\mathbf{m}} t'\ ]}$$

$$(d, [\ \mathcal{T}^{\mathfrak{m}}_{d-1}(t')\ ||\ \rho : t \xrightarrow{d}{}^{*}_{\mathbf{m}} t'\ ])$$

Plan

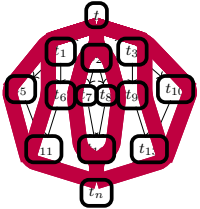
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Decrease proof

Main theorem

$$M \to_\beta N \qquad \Longrightarrow \qquad \mathcal{T}^{\mathbb{m}}(M) > \mathcal{T}^{\mathbb{m}}(N)$$

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Greater redexes $>$ For each target below, greater lower measure above

$$\begin{array}{ccc} t & & \mathcal{T}^{\mathfrak{m}}(t) = [..., (3, [\mathcal{T}^{\mathfrak{m}}_2(t_1), ..., \mathcal{T}^{\mathfrak{m}}_2(t_n)]), ...] \\ \mathbf{m} \big\downarrow j & & \\ t' & & \mathcal{T}^{\mathfrak{m}}(t') = [..., (3, [\mathcal{T}^{\mathfrak{m}}_2(t'_1), ..., \mathcal{T}^{\mathfrak{m}}_2(t'_m)]), ...] \end{array}$$

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Similar redexes $=$ Upwards injection $+ 1$

$$t \quad \mathcal{T}^{\mathtt{m}}(t) = [..., (j, [\mathcal{T}^{\mathtt{m}}_{j-1}(t'), \mathcal{T}^{\mathtt{m}}_{j-1}(t_1), ..., \mathcal{T}^{\mathtt{m}}_{j-1}(t_n)]), ...]$$

$$\mathtt{m} \downarrow j$$

$$t' \quad \mathcal{T}^{\mathtt{m}}(t') = [..., (j, [\mathcal{T}^{\mathtt{m}}_{j-1}(t_1), ..., \mathcal{T}^{\mathtt{m}}_{j-1}(t_n)]), ...]$$

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Little redexes $>$ Non-erasure

$$t \;\Big|_{\mathtt{m}}\; j \;\downarrow\; t'$$

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Quick conclusions **How redex degrees could be *complemented* to make the DM strong**

- *Easeness* dimensions
- Reducibility "opposite"
- Future distillation

BONUS!

$\mathbf{RED}_{\sim}$: Reducibility candidates modulo isomorphisms

ongoing work with Díaz-Caro

RED$_\sim$: Reducibility candidates modulo isomorphisms

$A \to (B \times C) \equiv (A \to B) \times (A \to C)$

$\lambda x.(x, x) \equiv (\lambda x.x, \lambda x.x)$

$(\lambda x.(x\,x))\,t \Longrightarrow ((\lambda x.x, \lambda x.x))\,t$

RED

$[\![A \to B]\!]_\rho$: must resist app

$A \to B \equiv \forall X.C$ must resist type app

$[\![A]\!]$ must resist eliminators of the whole class

RED$_\sim$: Reducibility candidates modulo isomorphisms

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$\to$ elim — $\times$ intro

neutrality unstable

RED

$[\![A \to B]\!]_\rho$: must resist app

$A \to B \equiv \forall X.C$ — must resist type app

$[\![A]\!]$ — must resist eliminators of the whole class

no CR3 for us

Inductive case using CR3:

instead of $(\lambda x.t)s \in [\![B]\!]$

we test $(\lambda x.t)\alpha_1 \dots \alpha_n \in SN_x$

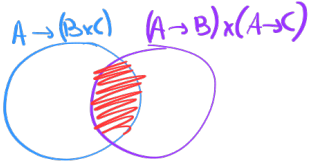
battering all $\Rightarrow$ elims of $[\![X]\!]_\rho$

Adapting Parigot's inductive candidates [Parigot'97]

$[\![X]\!]_\rho$ gives rise to mp on elems $\Longrightarrow$ Parigot's inductive def. of candidates

Candidate = derivation tree

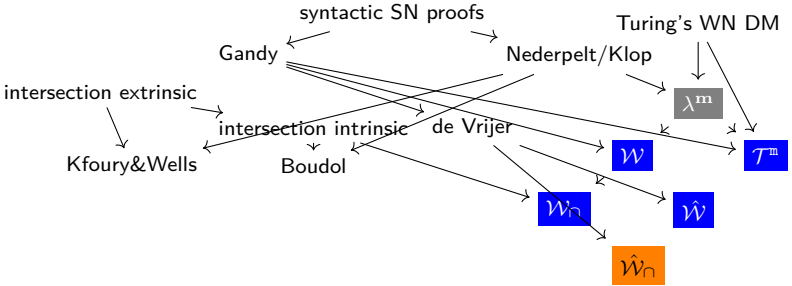
$\rho(X)$ is a tree we can get mps from



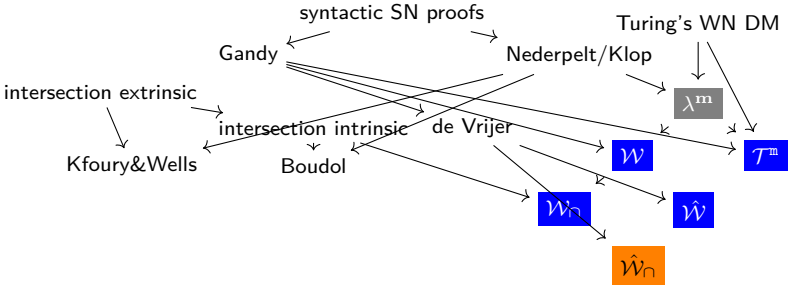
I keep the terms that reast b+h

Conclusions and future work

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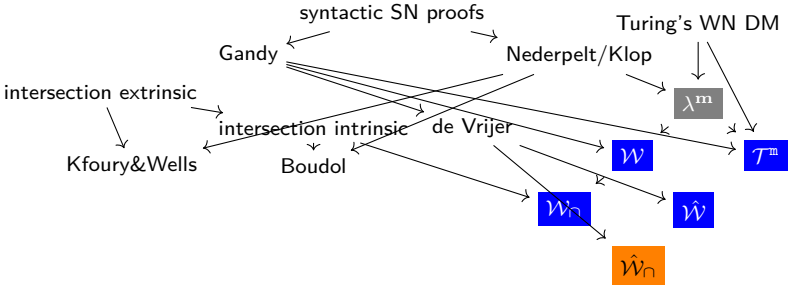


Conclusions and future work



- New elements
- New execution-decreasing info "inside" terms
- New techniques for proving SN
- Following a fully syntactic approach relying on $\lambda^{\mathbb{m}}$

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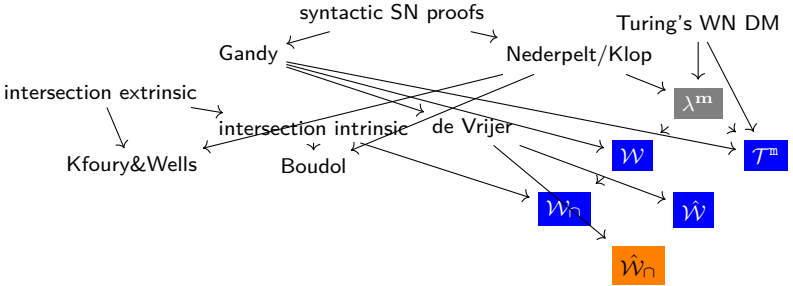
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Depth path

more insight from $\mathcal{W}$ and $\mathcal{T}^{\mathtt{m}}$

- exact $\hat{\mathcal{W}}_\cap$
- simplified $\mathcal{T}^{\mathtt{m}}$

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Expansion path

adapting $\mathcal{W}$ to other systems

- HN $\Lambda_\cap$
- $\Lambda_{\equiv\setminus\mathsf{iso}}$
- System F

Thanks!