Strong normalization through idempotent intersection types: a new syntactical approach

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Motivation

To find *simpler* proofs of *strong normalization* for idempotent intersection types

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Strong normalization proof techniques

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Semantic approach: reducibility candidates/logical relations [Tait'67,Girard'72]

- Define a **denotational semantic** for types based on **termination**
- **Prove soundness of typed terms** w.r.t. the semantics

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- Define a **denotational semantic** for types based on **termination**
- **Prove soundness of typed terms** w.r.t. the semantics

Syntactic approach: decreasing measures [Gandy'80, de Vrijer'87]

- Define a **mapping** from terms to a **well founded order**
- Such that it **decreases along reduction**
- TLCA Problem#26 (for STLC, posed by Gödel)

Results

1. A **decreasing measure** based on enriching the calculus with memories

Definition

A mapping

$$\# : \Lambda \to \mathit{WFO}$$

satisfying

$$M \to_\beta N \implies \#(M) > \#(N)$$

Corollary

$$\nexists \quad M_1 \to_\beta M_2 \to_\beta \cdots$$

$$\#(M_1) > \#(M_2) > \cdots$$

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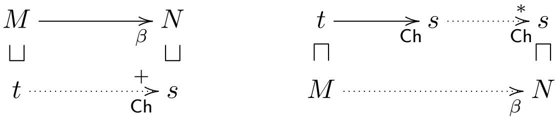
$$\#(M_1) > \#(M_2) > \cdots$$

2. An intrinsically typed (*i.e.* **à la Church**) version of **idempotent intersection types**
   - usually presented à la Curry
   - both systems **simulate** each other



Idempotent Intersection Types ($\Lambda_{\cap}^{\mathsf{Cu}}$)

[Coppo-Dezzani'79]

Key idea Allowing variables to have multiple types

$$\Gamma, x : A \vdash x : A \qquad \rightsquigarrow \qquad \Gamma, x : \{A_1, ..., A_n\} \vdash x : A_i$$

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Grammar of types

$$\begin{array}{lcll} A & ::= & a \mid \vec{A} \to A & (\vec{A} \neq \varnothing) \\ \vec{A} & ::= & \{A_1, ..., A_n\} & (A_i \neq A_j \text{ if } i \neq j) \quad (n \in \mathbb{N}) \end{array}$$

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Typing rules

$$\frac{B \in \vec{A}}{\Gamma, x : \vec{A} \vdash_{\mathsf{e}} x : B}\ \texttt{e-var} \qquad \frac{(\Gamma \vdash_{\mathsf{e}} N : A_i)_{i \in I} \quad A_i \neq A_j \text{ if } i \neq j}{\Gamma \Vdash_{\mathsf{e}} N : \{A_1, \ldots, A_n\}}\ \texttt{e-many}$$

$$\frac{\Gamma, x : \vec{A} \vdash_{\mathsf{e}} M : B}{\Gamma \vdash_{\mathsf{e}} \lambda x.M : \vec{A} \to B}\ \texttt{e-I}{\to} \qquad \frac{\Gamma \vdash_{\mathsf{e}} M : \vec{A} \to B \quad \Gamma \Vdash_{\mathsf{e}} N : \vec{A} \quad \vec{A} \neq \varnothing}{\Gamma \vdash_{\mathsf{e}} M\, N : B}\ \texttt{e-E}{\to}$$

Example

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Typing a self–application

$$\dfrac{\dfrac{\dfrac{x : \{A \to A, A\} \vdash_{\mathsf{Cu}} x : A \to A \quad x : \{A \to A, A\} \vdash_{\mathsf{Cu}} x : A}{x : \{A \to A, A\} \vdash_{\mathsf{Cu}} xx : A}}{\vdash_{\mathsf{Cu}} \lambda x.xx : \{A \to A, A\} \to A} \quad \dfrac{\vdash_{\mathsf{Cu}} \lambda x.x : A \to A \quad \vdash_{\mathsf{Cu}} \lambda x.x : A}{\Vdash_{\mathsf{Cu}} \lambda x.x : \{A \to A, A\}}}{\vdash_{\mathsf{Cu}} (\lambda x.xx)(\lambda x.x) : A}$$

Part I: an intrinsically typed idempotent intersection system

Idempotent Intersection Types: a Church presentation ($\Lambda_\cap^{\mathsf{Ch}}$)

Why?

- The measure technique is **based on redex degrees** ($\therefore$ on types of subterms)
- So: We need to **handle derivations**
- But: The technique requires **syntactical "intermediate" derivations**
- And: these are **not representable** in the presentation **à la Curry**

À la Curry

$$\dfrac{\dfrac{\dfrac{x : \{A \to A, A\} \vdash_{\mathsf{Cu}} x : A \to A \quad x : \{A \to A, A\} \vdash_{\mathsf{Cu}} x : A}{x : \{A \to A, A\} \vdash_{\mathsf{Cu}} xx : A}}{\vdash_{\mathsf{Cu}} \lambda x.xx : \{A \to A, A\} \to A} \quad \dfrac{\vdash_{\mathsf{Cu}} \lambda x.x : A \to A \quad \vdash_{\mathsf{Cu}} \lambda x.x : A}{\Vdash_{\mathsf{Cu}} \lambda x.x : \{A \to A, A\}}}{\vdash_{\mathsf{Cu}} (\lambda x.xx)(\lambda x.x) : A}$$

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A linearization inspired by Kfoury's

$$\dfrac{(\Gamma \vdash_{\mathsf{Cu}} N : A_i)_{i \in 1..n} \quad \ldots}{\Gamma \Vdash_{\mathsf{Cu}} N : \{A_1, \ldots, A_n\}} \quad \Longrightarrow \quad \dfrac{(\Gamma \vdash_{\mathsf{Ch}} s_i : A_i)_{i \in 1..n} \qquad A_i \neq A_j \text{ if } i \neq j}{\Gamma \Vdash_{\mathsf{Ch}} \{s_1, \ldots, s_n\} : \{A_1, \ldots, A_n\}}$$

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À la Church

$$\rightsquigarrow \qquad (\lambda x^{\{\{A\} \to A, A\}}.x^{\{A\} \to A} x^A)\{\lambda x^{\{A\} \to A}.x, \lambda x^A.x\}$$

Idempotent Intersection Types: a Church presentation ($\Lambda_{\cap}^{\mathsf{Ch}}$)

Substitution

à la Curry

$$(\lambda x.\, x\ x)\ (\lambda x.x)$$
$$\to_\beta$$
$$(\lambda x.x)\ (\lambda x.x)$$

à la Church

$$(\lambda x^{\{A \to A, A\}}.\, x^{A \to A}\, x^{A})\{\lambda x^{A \to A}.x,\ \lambda x^{A}.x\}$$
$$\to_{\mathsf{Ch}}$$
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Substitution

à la Curry

$$(\lambda x. x\ x)(\lambda x.x) \to_\beta (\lambda x.x)\ (\lambda x.x)$$

$$(\lambda x.t)s \to_\beta t[s/x]$$

$\rightsquigarrow$

à la Church

$$(\lambda x^{\{A\to A, A\}}. x^{A\to A}, x^{A})\{\lambda x^{A\to A}.x, \lambda x^{A}.x\} \to_{\mathsf{Ch}} (\lambda x^{A\to A}.x)\ (\lambda x^{A}.x)$$

$$(\lambda x^{\vec{A}}.t)\vec{s} \to_{\mathsf{Ch}} t[s_1/x^{A_1}, \ldots, s_n/x^{A_n}]$$

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Substitution

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$$(\lambda x. x \ x)(\lambda x.x) \to_\beta (\lambda x.x) \ (\lambda x.x)$$

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$$(\lambda x^{\{A\to A, A\}}. x^{A\to A}, x^{A})\{\lambda x^{A\to A}.x, \lambda x^{A}.x\} \to_{\mathsf{Ch}} (\lambda x^{A\to A}.x)(\lambda x^{A}.x)$$

$$(\lambda x.t)s \to_\beta t[s/x] \quad \rightsquigarrow \quad (\lambda x^{\vec{A}}.t)\vec{s} \to_{\mathsf{Ch}} t[s_1/x^{A_1}, \ldots, s_n/x^{A_n}]$$

Bijection between a set–term and its set–type

$\Rightarrow$) given $s' \in \vec{s}$, it has type some type A' by `i-many`, which is **unique** (à la Church typing)

$\Leftarrow$) given $A' \in \vec{A}$, **by injectivity** ($A_i \neq A_j$ if $i \neq j$) there is **only one derivation** $\Gamma \vdash_{\mathsf{Ch}} s' : A'$

Relating $\Lambda_\cap^{\mathsf{Cu}}$ and $\Lambda_\cap^{\mathsf{Ch}}$

Reduction difference Inside the argument of an application

$$t\{s_1, s_2, \ldots, s_n\} \to_{\mathsf{Ch}} t\{s'_1, s_2, \ldots, s_n\} \to_{\mathsf{Ch}} t\{s'_1, s'_2, \ldots, s_n\} \to_{\mathsf{Ch}} \ldots \to_{\mathsf{Ch}} t\{s'_1, s'_2, \ldots, s'_n\}$$

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Relating terms and derivations

Uniformity set–term with "equal" subterms

Refinement relate uniform set–terms and terms

$$x^D\{\, I^A I^B \,,\, I^B I^C \,\} \quad \sqsubset \quad \frac{x^D \sqsubset x \qquad (I^A I^B)^{\mathsf{e}} = (I^C I^D)^{\mathsf{e}} = II}{x(II)}$$

$$x^D\{\, I^A I^B \,,\, I^C \,\} \qquad x^D\{\, I^B \,,\, I^B I^C \,\}$$

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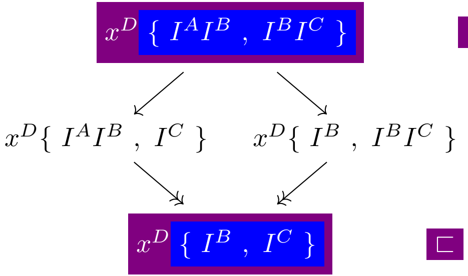

$$x^D\{\ I^A I^B\ ,\ I^B I^C\ \} \;\sqsubset\; \frac{x^D \sqsubset x \qquad (I^A I^B)^{\mathsf{e}} = (I^C I^D)^{\mathsf{e}} = II}{x(II)}$$

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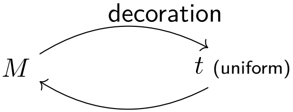
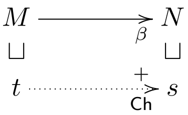
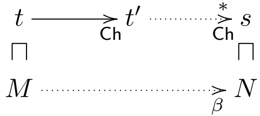
Correspondence



Simulation



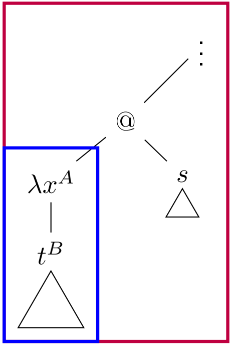


Part II: a decreasing measure for $\Lambda_{\cap}^{\mathsf{Ch}}$

Redex degrees

Definition

degree of the type of its abstraction



$deg(A \to B)$

Redex degrees

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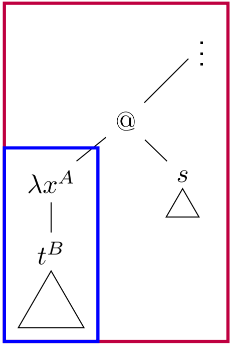
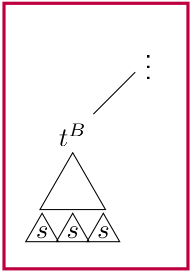


$deg(A \to B)$

$\to_\beta$

Turing's observation

a redex contraction can only create smaller degree redexes



$deg(R \text{ new}) < deg(A \to B)$

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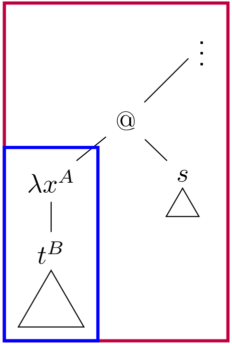

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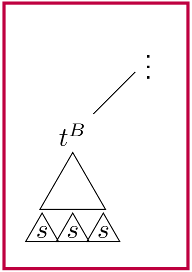

$deg(R \text{ new}) < deg(A \to B)$

Redex creation [Lévy, 1978]

identity applied to a λ	$(\lambda x.x)\,(\lambda y.t)\,s$	$\to_\beta$	$(\lambda y.t)\,s$
λ body is a λ	$(\lambda x.\lambda y.t)\,s\,u$	$\to_\beta$	$(\lambda y.t[s/x])\,u$
replaced var in app position	$(\lambda x.\ldots x\,s\ldots)\,(\lambda y.t)$	$\to_\beta$	$\ldots(\lambda y.t)\,s[\lambda y.t/x]\ldots$

Redex degrees

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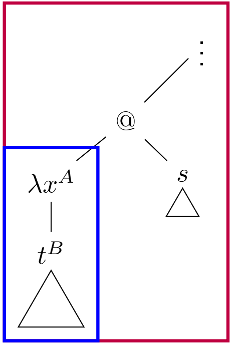

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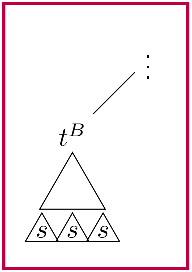

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identity applied to a λ	$(\lambda x^{B\to C}.x)\,(\lambda y^B.t^C)\,s^B$	$\to_\beta$	$(\lambda y^B.t^C)\,s^B$
λ body is a λ	$(\lambda x^A.\lambda y^B.t^C)\,s^A\,u^B$	$\to_\beta$	$(\lambda y^B.t^C[s^A/x])\,u^B$
replaced var in app position	$(\lambda x^{A\to B}.\dots x\,s^A\dots)\,(\lambda y^A.t^B)$	$\to_\beta$	$\dots(\lambda y^A.t^B)\,s^A[\lambda y^A.t^B/x]\dots$

The memory $\Lambda_{\cap}^{\mathsf{Ch}}$

Definition

$$t \; ::= \; x^{\vec{A}} \mid \lambda x.t \mid t\,\vec{t} \mid t\langle\vec{t}\rangle \qquad (\lambda x^{\vec{A}}.t)\vec{s} \to_m t[\vec{s}/x^{\vec{A}}]\langle\vec{s}\rangle$$

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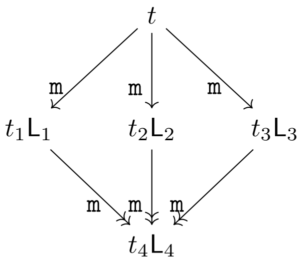
Why not to erase?

- **Retain information**



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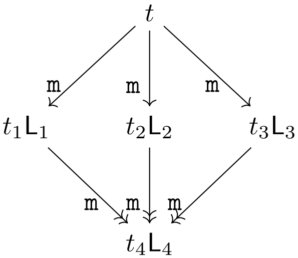

WCR

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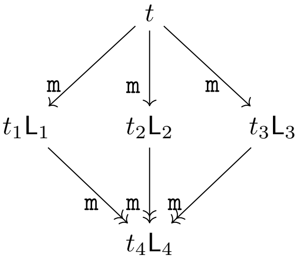
Example

Let $A = \{a\} \to a$, $B = \{A\} \to A$ and $C = \{B\} \to B$:



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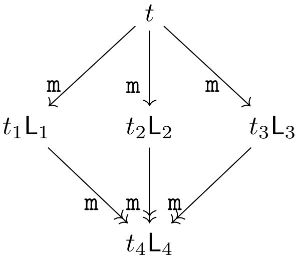

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$$(\lambda x^{\{B,A\}}.x^B x^A)\{I^C I^B, I^B I^A\}$$

WCR

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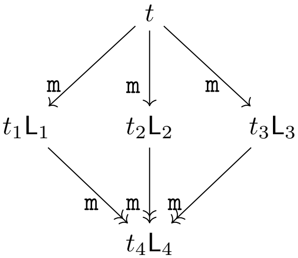

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WCR

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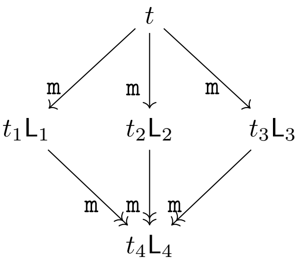

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Why not to erase?

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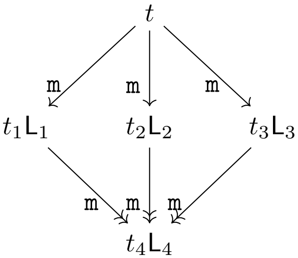

Example

Let $A = \{a\} \rightarrow a$, $B = \{A\} \rightarrow A$ and $C = \{B\} \rightarrow B$:

$$(\lambda x^{\{B,A\}}.x^B x^A)\{I^C I^B, I^B I^A\}$$

$$\rightarrow_m \quad (\lambda x^{\{B,A\}}.x^B x^A)\{I^B\langle I^B\rangle, I^B I^A\}$$

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$$\rightarrow_m \quad (I^B\langle I^B\rangle)(I^A\langle I\rangle^A)\,\langle\{I^B\langle I^B\rangle, I^A\langle I^A\rangle\}\rangle$$

WCR

The memory $\Lambda_{\cap}^{\mathsf{Ch}}$

Definition

$$t \ ::= \ x^{\vec{A}} \mid \lambda x.t \mid t\,\vec{t} \mid t\langle\vec{t}\rangle \qquad (\lambda x^{\vec{A}}.t)\vec{s} \to_m t[\vec{s}/x^{\vec{A}}]\langle\vec{s}\rangle$$

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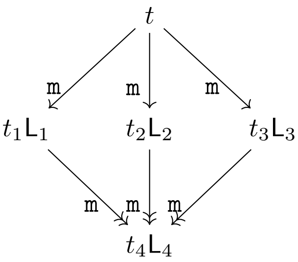

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$\mathcal{W}_\cap$-measure: definition

Operations

- **weight** of a term:
  $\mathsf{w}(t)$ = amount of memories

e.g. $I\langle I\rangle\langle I\langle I\rangle\rangle\ \langle\{I^2\langle I^2\rangle, I\langle I\rangle\}\rangle = 6$

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- **simplification** of a term for degree d:
  $\mathsf{S}_d(t)$ = "bottom-up" "contraction" of all d redexes $\quad$ (def. by structural recursion)

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- **full simplification**

$$\mathsf{S}_*(t) = \mathsf{S}_1(\ldots \mathsf{S}_{\mathsf{maxdeg}}(t)\ldots)$$

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$\mathcal{W}_\cap$-measure

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$\mathcal{W}_\cap$-measure: proof

Properties

- Reduction arrives at simplification $t \to_m^* \mathsf{S}_*(t)$

$\mathcal{W}_\cap$-measure: proof

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- Max–degree simplification decreases max–degree $D_{max}(t) > D_{max}(\mathsf{S}_{D_{max}}(t))$

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Theorem: $\mathcal{W}_\cap$ decreases along reduction

$$t \to_{\mathsf{Ch}} s \quad \Longrightarrow \quad \mathcal{W}_\cap(t) > \mathcal{W}_\cap(s)$$

$\mathcal{W}_\cap$-measure: proof

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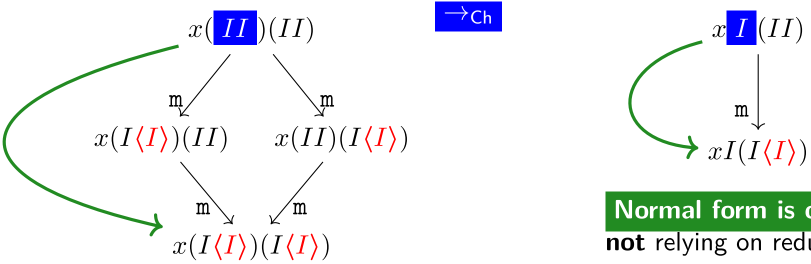
Intuitively



Normal form is obtained through S_*
not relying on reduction

Part III: a decreasing measure for $\Lambda_{\cap}^{\mathbf{Cu}}$

Lifting the result

$\mathcal{W}_\cap^{\mathsf{Cu}}$-measure

- To define $\mathcal{W}_\cap^{\mathsf{Ch}}$, we take the **weight** of the **full simplification** of t
- To define $\mathcal{W}_\cap^{\mathsf{Cu}}$, we **decorate** M with types, and then we apply $\mathcal{W}_\cap^{\mathsf{Ch}}$

$$\mathcal{W}_\cap^{\mathsf{Cu}}(M) = \mathsf{w}(\mathsf{S}_*(M^{\mathsf{Ch}}))$$

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$\mathcal{W}_\cap^{\mathsf{Cu}}$ decreases along reduction

$$M \to N \implies \mathcal{W}_\cap^{\mathsf{Cu}}(M) > \mathcal{W}_\cap^{\mathsf{Cu}}(N)$$

$$\begin{array}{ccc}
\boldsymbol{M} & \to_\beta & \boldsymbol{N} \\
\sqcup & & \sqcup \\
t & \to_{\mathsf{Ch}}^{+} & s \\
| & & | \\
\mathcal{W}_\cap(t) & & \mathcal{W}_\cap(s)
\end{array}$$

Lifting the result

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$\mathcal{W}_{\cap}^{\mathsf{Cu}}$ decreases along reduction $M \to N \implies \mathcal{W}_{\cap}^{\mathsf{Cu}}(M) > \mathcal{W}_{\cap}^{\mathsf{Cu}}(N)$

$$\begin{array}{ccc}
\boldsymbol{M} & \to_{\boldsymbol{\beta}} & \boldsymbol{N} \\
\sqcup & & \sqcup \\
t & \to^{+}_{\mathsf{Ch}} & s \\
| & & | \\
\mathcal{W}_{\cap}(t) & & \mathcal{W}_{\cap}(s)
\end{array}$$

$\Lambda_{\cap}^{\mathsf{Cu}}$ is strongly normalizing $\Gamma \vdash_{\mathsf{Cu}} M : A \implies M \in SN$

The converse

$$M \in SN \;\; \begin{array}{l} \diagup \; (M^{\mathsf{Ch}})^{\mathsf{e}} \\ \qquad \sqcup \\ \diagdown \; M^{\mathsf{Ch}} \end{array}$$

$\Lambda_{\cap}^{\mathsf{Cu}}$ is complete for strong normalization $M \in SN \implies \Gamma \vdash_{\mathsf{Cu}} M : A$

Proof

- By induction on the inductive definition of SN
- Using:
  - refinement
  - a head subject expansion lemma

Related works

Existing decreasing measures for idempotent intersection types

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[Kfoury & Wells'95]

- **Domain of DM:** **multiset of natural numbers**
- **Methodology:** **indirect**, *i.e.* decreases for specific strategy
- **Base calculus:** à la Church, ad hoc

[Boudol'03]

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Related works

Existing decreasing measures for idempotent intersection types

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Our proposal

- **Domain of DM:** **natural number**
- **Methodology:** DM proving SN **(direct)**
- **Base calculus:** à la Church, **proven *in simulation*** with Curry

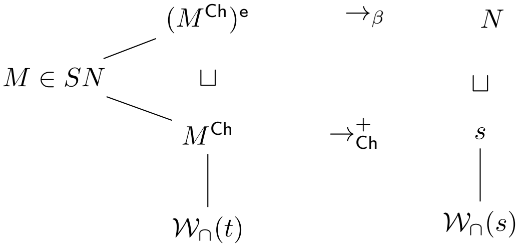
Conclusion

- We defined a Church version of idempotent intersection types
- We defined a decreasing measure for the Church version, and extend it to the Curry version
- The defined measure is simpler than the previous ones
- We successfully extended the technique from STLC

Future work

- **Refinement** of the measure **to exactness**, *i.e.* such that $\mathcal{W}_{\cap}(M)$ is the amount of reduction steps of the longest reduction chain
- **Adaptation** of the technique to the idempotent intersection type system characterizing ***head normalization***
- **Adaptation of the original technique** to other complex systems
- **Formalization** in proof assistants
- **Compare** in depth the original technique to Gandy's and de Vrijer's

Conclusion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