Strong normalization through idempotent intersection types: a new syntactical approach

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Motivation

To find *simpler* proofs of *strong normalization* for idempotent intersection types

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Strong normalization proof techniques

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Semantic approach: reducibility candidates/logical relations [Tait'67,Girard'72]

- Define a **denotational semantic** for types based on **termination**
- **Prove soundness of typed terms** w.r.t. the semantics

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- Define a **denotational semantic** for types based on **termination**
- **Prove soundness of typed terms** w.r.t. the semantics

Syntactic approach: decreasing measures [Gandy'80, de Vrijer'87]

- Define a **mapping** from terms to a **well founded order**
- Such that it **decreases along reduction**
- TLCA Problem#26 (for STLC, posed by Gödel)

Results

1. A **decreasing measure** based on enriching the calculus with memories

Definition

A mapping

$$\# : \Lambda \to \mathit{WFO}$$

satisfying

$$M \to_\beta N \implies \#(M) > \#(N)$$

Corollary

$$\nexists \quad M_1 \to_\beta M_2 \to_\beta \cdots$$

$$\#(M_1) > \#(M_2) > \cdots$$

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2. An intrinsically typed (*i.e.* **à la Church**) version of **idempotent intersection types**
   - usually presented à la Curry
   - both systems **simulate** each other

$$\begin{array}{ccc} M & \xrightarrow[\beta]{} & N \\ \sqcup & & \sqcup \\ t & \overset{+}{\underset{\mathsf{Ch}}{\dashrightarrow}} & s \end{array} \qquad\qquad \begin{array}{ccccc} t & \xrightarrow[\mathsf{Ch}]{} & s & \overset{*}{\underset{\mathsf{Ch}}{\dashrightarrow}} & s \\ \sqcap & & & & \sqcap \\ M & & \underset{\beta}{\dashrightarrow} & & N \end{array}$$

Idempotent Intersection Types ($\Lambda_{\cap}^{\mathsf{Cu}}$)

[Coppo-Dezzani'79]

Grammar of types

$$\begin{array}{lcll} A & ::= & a \mid \vec{A} \to A & (\vec{A} \neq \varnothing) \\ \vec{A} & ::= & \{A_1, \ldots, A_n\} & (A_i \neq A_j \text{ if } i \neq j) \quad (n \in \mathbb{N}) \end{array}$$

Typing rules

$$\frac{B \in \vec{A}}{\Gamma, x : \vec{A} \vdash_{\mathsf{e}} x : B}\ \texttt{e-var} \qquad \frac{(\Gamma \vdash_{\mathsf{e}} N : A_i)_{i \in I} \quad A_i \neq A_j \text{ if } i \neq j}{\Gamma \Vdash_{\mathsf{e}} N : \{A_1, \ldots, A_n\}}\ \texttt{e-many}$$

$$\frac{\Gamma, x : \vec{A} \vdash_{\mathsf{e}} M : B}{\Gamma \vdash_{\mathsf{e}} \lambda x.M : \vec{A} \to B}\ \texttt{e-I}{\to} \qquad \frac{\Gamma \vdash_{\mathsf{e}} M : \vec{A} \to B \quad \Gamma \Vdash_{\mathsf{e}} N : \vec{A} \quad \vec{A} \neq \varnothing}{\Gamma \vdash_{\mathsf{e}} M\,N : B}\ \texttt{e-E}{\to}$$

Part I: an intrinsically typed idempotent intersection system

Idempotent Intersection Types: a Church presentation ($\Lambda_{\cap}^{\mathsf{Ch}}$)

Why?

- The measure technique is **based on redex degrees** ($\therefore$ on types of subterms)
- So: We need to **handle derivations**
- But: The technique requires **syntactical "intermediate" derivations**
- And: these are **not representable** in the presentation **à la Curry**

À la Curry

$$\dfrac{\dfrac{\dfrac{x : \{A \to A, A\} \vdash_{\mathsf{Cu}} x : A \to A \quad x : \{A \to A, A\} \vdash_{\mathsf{Cu}} x : A}{x : \{A \to A, A\} \vdash_{\mathsf{Cu}} xx : A}}{\vdash_{\mathsf{Cu}} \lambda x.xx : \{A \to A, A\} \to A} \quad \dfrac{\vdash_{\mathsf{Cu}} \lambda x.x : A \to A \quad \vdash_{\mathsf{Cu}} \lambda x.x : A}{\Vdash_{\mathsf{Cu}} \lambda x.x : \{A \to A, A\}}}{\vdash_{\mathsf{Cu}} (\lambda x.xx)(\lambda x.x) : A}$$

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A linearization inspired by Kfoury's

$$\dfrac{(\Gamma \vdash_{\mathsf{Cu}} N : A_i)_{i \in 1..n} \quad \ldots}{\Gamma \Vdash_{\mathsf{Cu}} N : \{A_1, \ldots, A_n\}} \qquad \Longrightarrow \qquad \dfrac{(\Gamma \vdash_{\mathsf{Ch}} s_i : A_i)_{i \in 1..n} \qquad A_i \neq A_j \text{ if } i \neq j}{\Gamma \Vdash_{\mathsf{Ch}} \{s_1, \ldots, s_n\} : \{A_1, \ldots, A_n\}}$$

Idempotent Intersection Types: a Church presentation ($\Lambda_\cap^{\mathsf{Ch}}$)

À la Curry

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À la Church

$$\rightsquigarrow \qquad (\lambda x^{\{\{A\} \to A, A\}}.x^{\{A\} \to A} x^A)\{\lambda x^{\{A\} \to A}.x, \lambda x^A.x\}$$

Idempotent Intersection Types: a Church presentation ($\Lambda_{\cap}^{\mathsf{Ch}}$)

Substitution

à la Curry

$$(\lambda x. x\ x)\ (\lambda x.x)$$
$$\to_\beta$$
$$(\lambda x.x)\ (\lambda x.x)$$

à la Church

$$(\lambda x^{\{A\to A, A\}}. x^{A\to A}, x^{A})\{\lambda x^{A\to A}.x, \lambda x^{A}.x\}$$
$$\to_{\mathsf{Ch}}$$
$$(\lambda x^{A\to A}.x)\ (\lambda x^{A}.x)$$

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$$\to_{\mathsf{Ch}}$$
$$(\lambda x^{A\to A}.x)\ (\lambda x^{A}.x)$$

$$(\lambda x.t)s \to_\beta t[s/x] \quad \rightsquigarrow \quad (\lambda x^{\vec{A}}.t)\vec{s} \to_{\mathsf{Ch}} t[s_1/x^{A_1}, \ldots, s_n/x^{A_n}]$$

Idempotent Intersection Types: a Church presentation ($\Lambda_\cap^{\mathsf{Ch}}$)

Substitution

à la Curry		**à la Church**
$(\lambda x. x\ x)(\lambda x.x)$		$(\lambda x^{\{A\to A, A\}}. x^{A\to A}, x^{A})\{\lambda x^{A\to A}.x, \lambda x^{A}.x\}$
$\to_\beta$		$\to_{\mathsf{Ch}}$
$(\lambda x.x)\ (\lambda x.x)$		$(\lambda x^{A\to A}.x)\ (\lambda x^{A}.x)$
$(\lambda x.t)s \to_\beta t[s/x]$	$\rightsquigarrow$	$(\lambda x^{\vec{A}}.t)\vec{s} \to_{\mathsf{Ch}} t[s_1/x^{A_1}, \ldots, s_n/x^{A_n}]$

Bijection between a set–term and its set–type

$\Rightarrow$) given $s' \in \vec{s}$, it has type some type A' by `i-many`, which is **unique** (à la Church typing)

$\Leftarrow$) given $A' \in \vec{A}$, **by injectivity** ($A_i \neq A_j$ if $i \neq j$) there is **only one derivation** $\Gamma \vdash_{\mathsf{Ch}} s' : A'$

Relating $\Lambda_\cap^{\mathsf{Cu}}$ and $\Lambda_\cap^{\mathsf{Ch}}$

Reduction difference Inside the argument of an application

$$t\{s_1, s_2, \ldots, s_n\} \to_{\mathsf{Ch}} t\{s'_1, s_2, \ldots, s_n\} \to_{\mathsf{Ch}} t\{s'_1, s'_2, \ldots, s_n\} \to_{\mathsf{Ch}} \ldots \to_{\mathsf{Ch}} t\{s'_1, s'_2, \ldots, s'_n\}$$

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Relating terms and derivations

$$x^D\{\, I^A I^B \,,\, I^B I^C \,\} \quad \sqsubset \quad x(II)$$

$$x^D\{\, I^A I^B \,,\, I^C \,\} \qquad x^D\{\, I^B \,,\, I^B I^C \,\}$$

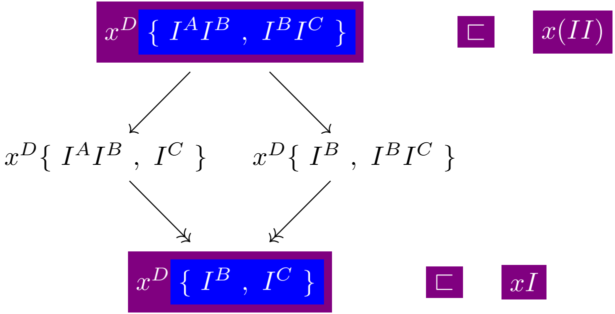
$$x^D\{\, I^B \,,\, I^C \,\} \quad \sqsubset \quad xI$$

Uniformity
set–term with "equal" subterms

Refinement
relate uniform set–terms with terms

Relating $\Lambda_{\cap}^{\mathsf{Cu}}$ and $\Lambda_{\cap}^{\mathsf{Ch}}$

Relating terms and derivations



Uniformity

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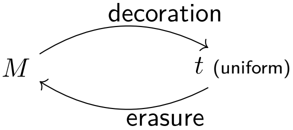
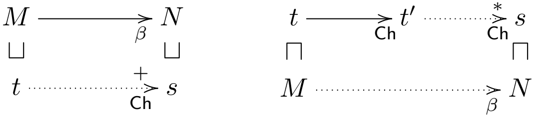
Refinement

relate uniform set–terms with terms

Correspondence



Simulation

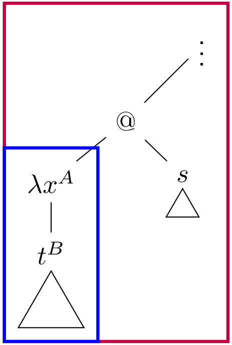


Part II: a decreasing measure for IIT

Redex degrees

Definition

degree of the type of its abstraction



deg$(A \to B)$

Redex degrees

Definition
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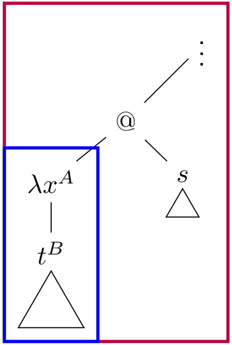
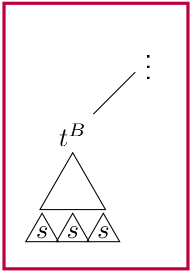


$deg(A \to B)$

$\to_\beta$

Turing's observation
a redex contraction can only create smaller degree redexes



$deg(R \text{ new}) < deg(A \to B)$

The memory $\Lambda_\cap^{\mathsf{Ch}}$

Definition

$$t \;::=\; x^{\vec{A}} \mid \lambda x.t \mid t\,\vec{t} \mid t\langle\vec{t}\rangle \qquad (\lambda x^{\vec{A}}.t)\vec{s} \rightarrow_m t[\vec{s}/x^{\vec{A}}]\langle\vec{s}\rangle$$

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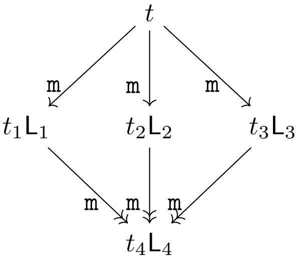
Why not to erase?

- **Retain information**



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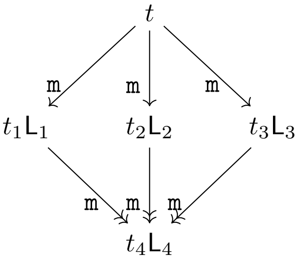

WCR

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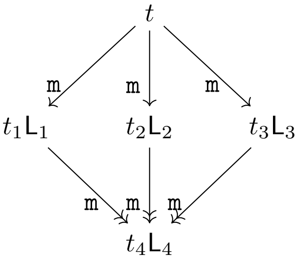

WCR

Example

Let $A = \{a\} \to a$, $B = \{A\} \to A$ and $C = \{B\} \to B$:

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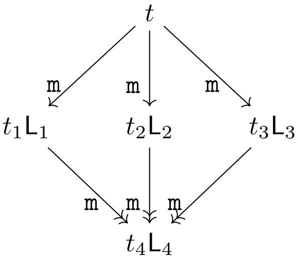

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$$(\lambda x^{\{B,A\}}.x^B x^A)\{I^C I^B, I^B I^A\}$$

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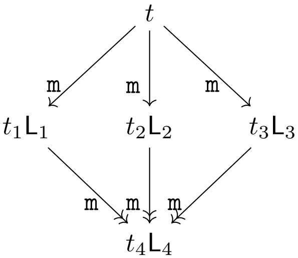

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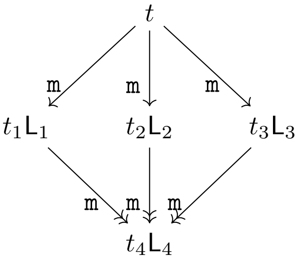

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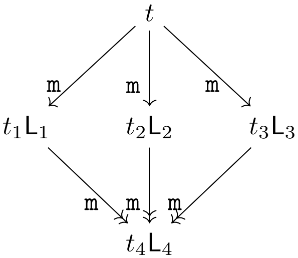

Example

Let $A = \{a\} \rightarrow a$, $B = \{A\} \rightarrow A$ and $C = \{B\} \rightarrow B$:

$$(\lambda x^{\{B,A\}}.x^B x^A)\{I^C I^B, I^B I^A\}$$

$$\rightarrow_m \quad (\lambda x^{\{B,A\}}.x^B x^A)\{I^B\langle I^B\rangle, I^B I^A\}$$

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$$\rightarrow_m \quad (I^B\langle I^B\rangle)(I\langle I\rangle)\,\langle\{I^B\langle I^B\rangle, I\langle I\rangle^A\}\rangle$$

WCR

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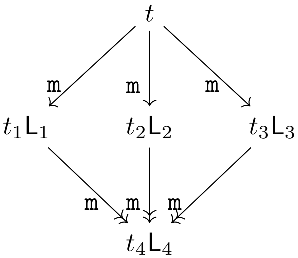

WCR

Example

Let $A = \{a\} \to a$, $B = \{A\} \to A$ and $C = \{B\} \to B$:

$$
\begin{aligned}
& (\lambda x^{\{B,A\}}.x^B x^A)\{I^C I^B, I^B I^A\} \\
\to_m \quad & (\lambda x^{\{B,A\}}.x^B x^A)\{I^B\langle I^B\rangle, I^B I^A\} \\
\to_m \quad & (\lambda x^{\{B,A\}}.x^B x^A)\{I^B\langle I^B\rangle, I\langle I\rangle^A\} \\
\to_m \quad & (I^B\langle I^B\rangle)(I\langle I\rangle)\langle\{I^B\langle I^B\rangle, I\langle I\rangle^A\}\rangle \\
\to_m \quad & I\langle I^A\rangle\langle I\langle I\rangle^A\rangle\ \langle\{I^B\langle I^B\rangle, I\langle I\rangle^A\}\rangle
\end{aligned}
$$

$\mathcal{W}_\cap$-measure: definition

Operations

- **weight** of a term:
  $\mathsf{w}(t)$ = amount of memories *e.g.* $I\langle I\rangle\langle I\langle I\rangle\rangle\ \langle\{I^2\langle I^2\rangle, I\langle I\rangle\}\rangle = 6$

$\mathcal{W}_\cap$-measure: definition

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- **simplification** of a term for degree d:
  $\mathsf{S}_d(t)$ = "bottom-up" "contraction" of all d redexes $\qquad$ (def. by structural recursion)

$$\mathsf{S}_d((\lambda x^{\vec{A}}.t')\mathtt{L}) \;=\; \mathsf{S}_d(t')\,[\,\mathsf{S}_d(\vec{s})\,/x^{\vec{A}}]\langle\,\mathsf{S}_d(\vec{s})\,\rangle\,\mathsf{S}_d(\mathtt{L}) \qquad \text{if it is of degree } d$$

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- **full simplification**

$$\mathsf{S}_*(t) = \mathsf{S}_1(\dots\mathsf{S}_{\mathsf{maxdeg}}(t)\dots)$$

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$\mathcal{W}_\cap$-measure

t

$\mathcal{W}_\cap$-measure: definition

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$\mathcal{W}_\cap$-measure

$t \qquad \mathsf{S}_*(t)$

$\mathcal{W}_\cap$-measure: definition

Operations

- **weight** of a term:
  $\mathsf{w}(t)$ = amount of memories *e.g.* $I\langle I\rangle\langle I\langle I\rangle\rangle\ \langle\{I^2\langle I^2\rangle, I\langle I\rangle\}\rangle = 6$

- **simplification** of a term for degree d:
  $\mathsf{S}_d(t)$ = "bottom-up" "contraction" of all d redexes (def. by structural recursion)

$$\mathsf{S}_d((\lambda x^{\vec{A}}.t')\mathtt{L}) = \mathsf{S}_d(t')[\mathsf{S}_d(\vec{s})/x^{\vec{A}}]\langle\mathsf{S}_d(\vec{s})\rangle\mathsf{S}_d(\mathtt{L})$$ if it is of degree d

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$$\mathcal{W}_\cap(t) \;=\; \mathsf{w}(\mathsf{S}_*(t))$$

$\mathcal{W}_{\cap}$-measure: proof

Properties

- Reduction arrives at simplification $t \rightarrow^*_m \mathsf{S}_*(t)$

$\mathcal{W}_\cap$-measure: proof

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$\mathcal{W}_\cap$-measure: proof

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$$t \to_{\mathsf{Ch}} s \quad \Longrightarrow \quad \mathcal{W}_\cap(t) > \mathcal{W}_\cap(s)$$

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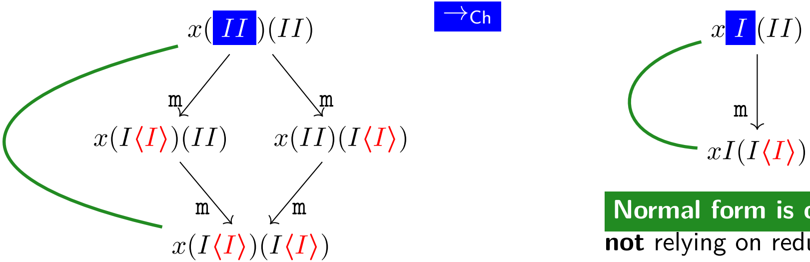
Intuitively



Normal form is obtained through S_*
not relying on reduction

Part III: Conclusion

Conclusion: lifting the result

$\mathcal{W}^{\mathsf{Cu}}_{\cap}$-measure $\mathcal{W}^{\mathsf{Cu}}_{\cap}(M) = \mathsf{w}(\mathsf{S}_*(M^{\mathsf{Ch}}))$

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$\mathcal{W}_\cap^{\mathsf{Cu}}$-measure $\mathcal{W}_\cap^{\mathsf{Cu}}(M) = \mathsf{w}(\mathsf{S}_*(M^{\mathsf{Ch}}))$

$\mathcal{W}_\cap^{\mathsf{Cu}}$ decreases along reduction $M \to N \implies \mathcal{W}_\cap^{\mathsf{Cu}}(M) > \mathcal{W}_\cap^{\mathsf{Cu}}(N)$

$$
\begin{array}{ccc}
\boldsymbol{M} & \to_\beta & \boldsymbol{N} \\
\sqcup & & \sqcup \\
t & \to^+_{\mathsf{Ch}} & s \\
| & & | \\
\mathcal{W}_\cap(t) & & \mathcal{W}_\cap(s)
\end{array}
$$

Conclusion: lifting the result

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$\Lambda_\cap^{\mathsf{Cu}}$ is strongly normalizing $\Gamma \vdash_{\mathsf{Cu}} M : A \implies M \in SN$

Related works

Existing decreasing measures for idempotent intersection types

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- **Methodology:** **indirect**, *i.e.* decreases for specific strategy
- **Base calculus:** à la Church, ad hoc

[Boudol'03]

- **Domain of DM:** **pair of natural numbers**
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Our proposal

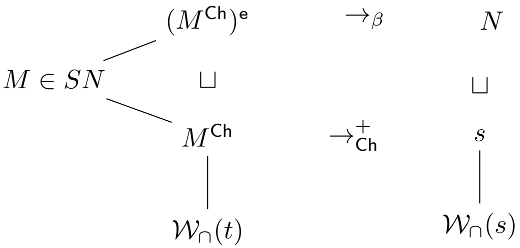
- **Domain of DM: natural number**
- **Methodology:** DM proving SN **(direct)**
- **Base calculus:** à la Church, **proven *in simulation*** with Curry

Future work

- **Refinement** of the measure **to exactness**, *i.e.* such that $\mathcal{W}_\cap(M)$ is the amount of reduction steps of the longest reduction chain

- **Adaptation** of the technique to the idempotent intersection type system characterizing ***head normalization***

Conclusion


$(M^{\mathsf{Ch}})^{\mathsf{e}}$
$\rightarrow_\beta$
N
$M \in SN$
$\sqcup$
$\sqcup$
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s
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