# A syntactic approach to Strong Normalization through decreasing measures

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# SN proofs: a brief overview

**Reducibility** [Tait'67, Girard'72]

- Interpretation of types into sets of *well-behaved* terms

$$[\![A \to B]\!] \;=\; \{\; t \;\mid\; \forall s \in [\![A]\!].\;\; ts \in [\![B]\!] \;\}$$

- Most widely known and used
- Concise
- Great adaptability

**Decreasing measures** [Gandy'80, de Vrijer'87]

**Definition**

A mapping $\# : \Lambda \to \mathit{WFO}$ satisfying $M \to_\beta N \implies \#(M) > \#(N)$

**Corollary**

$\nexists\; M_1 \to_\beta M_2 \to_\beta \cdots$

**Gandy and de Vrijer's** based on interpretations of $\boldsymbol{\Lambda}$ into **increasing functionals**

**Reduction of SN to WN** [Nederpelt'73, Klop'80]

- Through different flavours of $\lambda I$ ideas
- A WN proof

# Why?

**Why decreasing measures?**

- insight
- intuition
- metrics

**The koan #26**

- Posed by Gödel
- Submitted by Barendregt
- Find an "easy" mapping from $\lambda^{\rightarrow}$ to ordinals

**Why "syntactic"**

- sort of convention
- soft classification of SN proofs
- maybe **abstract** vs **concrete** would be better?
- **external** vs **internal**?
- we stick to the convention

**syntactic** = "internal" analysis over the **structure of terms** or the **rewriting relation**

| **semantic** | **syntactic** |
| --- | --- |
| reducibility (RC) | decreasing measures (DM) |
| | reduction of SN to WN (NK) |

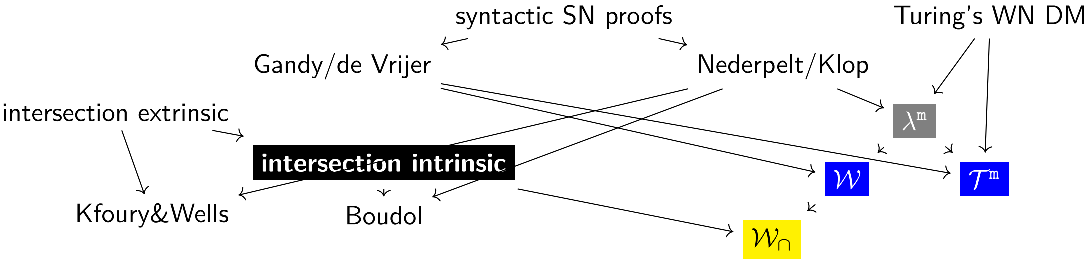
# Our work

**[Barenbaum & Sottile FSCD'23]**

- **An auxiliar calculus $\lambda^{\mathtt{m}}$** to manipulate (non-)erasure through memories
- **A simple measure $\mathcal{W}$** based on counting memories
- **A complex measure $\mathcal{T}^{\mathtt{m}}$** generalizing Turing's WN one

**[Work in progress with Barenbaum & Ronchi della Rocca]**

- A presentation of **idempotent intersection types a la Church**
- An adaptation of $\mathcal{W}$ to idempotent intersection types, $\mathcal{W}_\cap$



# The auxiliar non-erasing $\lambda^{\mathtt{m}}$-calculus

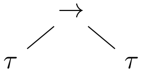
# Turing's measure: preliminary definitions

**Height of a type**

Length of longest path as tree

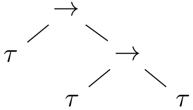
**Examples**

$\tau \to \tau$



$1$

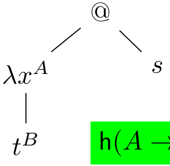
$\tau \to \tau \to \tau$



$2$

**Degree of a redex**

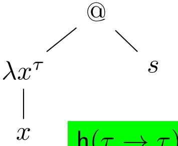
Height of its lambda



$\mathsf{h}(A \to B)$

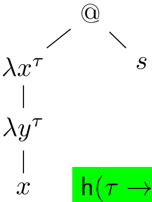
**Examples**

$(\lambda x^\tau.x)s$



$\mathsf{h}(\tau \to \tau) = 1$

$(\lambda x^\tau.\lambda y^\tau.x)s$



$\mathsf{h}(\tau \to \tau \to \tau) = 2$

# Turing's measure: Weak Normalization

**Map** terms $\mapsto$ multiset of the redex degrees $\mathcal{T}(M) = [\ d \mid R$ is a redex of degree $d$ in $M\ ]$

**Example**

$$\mathcal{T}(\underbrace{(\lambda x^\tau.\lambda y^\tau.x)\,\underbrace{(\lambda x^\tau.x)s}_{1}}_{2}) = [2,1]$$

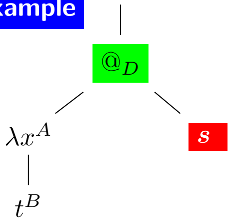
**Two crucial observations** [Turing, 1940s]

1. a redex cannot create redexes of greater or equal degree
2. a redex can copy redexes of any degree

**WN: choosing the redex to contract**

- has the greatest degree
- rightmost occurrence of that degree

**Example**




**Contracting rightmost greatest** $@_D$

- **cannot** create redexes $\geq D$
- **cannot** copy redexes $\geq D$

**Hence**

- one less $D$ redex

# The auxiliar non-erasing $\lambda^{\mathtt{m}}$−calculus

**Definition**

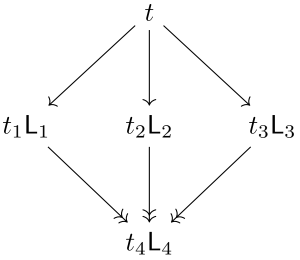
$t \ ::= \ x \mid \lambda x.t \mid t\,t \mid t\{t\} \qquad (\lambda x.t)s \to_m t[s/x]\{s\}$

**Properties**

- WCR
- WN
- SR

**Why not to erase?**

- Nederpelt-Klop's: *INC WCR WN* $\Rightarrow$ *DEC*
- **Retain information**




**Operations**

- **weight** of a term:
  $\mathsf{w}(t)$ = amount of memories
  *e.g.* $\mathsf{w}(x\{y\{z\}\}\{w\}) = 3$
- **simplification** of a term:
  $\mathsf{S}_D(t)$ = "bottom-up" contraction of all $D$ redexes
  $\mathsf{S}_*(t) = \mathsf{S}_1(\ldots \mathsf{S}_{\mathsf{maxdeg}}(t) \ldots)$

**Properties**

- Reduction arrives at simplification $\quad t \to_m^* \mathsf{S}_*(t)$
- Simplification is normal form $\quad \mathsf{S}_*(t) = \mathtt{nf}(t)$

# $\mathcal{W}$: counting memories

# Measure $\mathcal{W}$

**Recall** $(\lambda x.t)s \to_m t[s/x]\{s\}$ $\qquad$ $\mathsf{w}(t) =$ amount of memories

**Idea** $\qquad t \to s \quad \implies \quad \texttt{nf}(t)$ has more memories than $\texttt{nf}(s)$

**Definition**

$$\mathcal{W}(M) = \mathsf{w}(\mathsf{S}_*(M))$$

$$\begin{array}{ccccc} M & \twoheadrightarrow & \mathsf{S}_*(M) & \longmapsto & \mathsf{w}(\mathsf{S}_*(M)) \\ \downarrow & & & & \\ N & \twoheadrightarrow & \mathsf{S}_*(N) & \longmapsto & \mathsf{w}(\mathsf{S}_*(N)) \end{array}$$

**Theorem**

$$M \to_\beta N \quad \implies \quad \mathcal{W}(M) > \mathcal{W}(N)$$

# $\mathcal{T}^{\mathtt{m}}$: generalizing Turing's WN measure

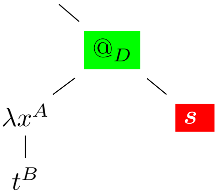
# Turing's measure: adaptation to SN

**Proposal** generalize the measure so that it decreases by contracting *any* redex

**Problems**

- $(>)$ A redex copies redexes of greater degree
- $(=)$ A redex copies redexes of same degree

**For instance**




with $\geq D$ redexes

**Idea**

$i)$ generalize $\mathcal{T}$ to a **family of measures $\mathcal{T}'_D$ indexed by a degree** $D \in \mathbb{N}$

$$\mathcal{T}'_2(M) = [2, 1] \qquad \text{and} \qquad \mathcal{T}'_1(M) = [1]$$

$ii)$ **associate extra information among with redex degrees**

*e.g.* consider smaller redexes' info (through the same measure)

$$\mathcal{T}'_2(M) = [\ (2, \mathcal{T}'_1(M)),\ (1, [])\ ] \qquad \mathcal{T}'_1(M) = [\ (1, [])\ ]$$

# Measure $\mathcal{T}^{\mathtt{m}}$

**More information...**

$$\mathcal{T}'_2(M) = [\ (2, ?),\ (1, ?)\ ]$$

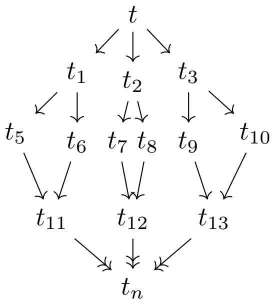
**Idea**

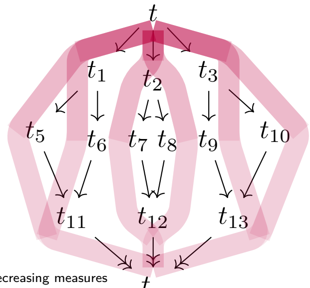
**Development of degree $D$**

reduction involving only redexes $D$

**All developments of degree $D$**

paths of the complete $D$-reduction graph from $t$



# $\mathcal{W}_\cap$:
# extending $\mathcal{W}$ to
# Idempotent Intersection Types

# Motivation

**Existing decreasing measures**

**[Kfoury & Wells'95]**

- **Domain of DM:** multiset of numbers
- **Methodology:** WN $\implies$ SN + DM proving WN (indirect)
- **Auxiliary calculus:** a la Curry

**[Boudol'03]**

- **Domain of DM:** pair of numbers
- **Methodology:** WN $\implies$ SN + DM proving WN (indirect)
- **Auxiliary calculus:** a la Church, ad hoc

**Our proposal** **Barenbaum, Ronchi della Rocca & Sottile (WIP)**

- **Domain of DM:** **number**
- **Methodology:** DM proving SN **(direct)**
- **Auxiliary calculus:** a la Church, **correspondent** of a la Curry calculus

# Idempotent Intersection Types (a la Curry)

**Key idea**

- Variables can have multiple types  *e.g.* $x : \{\tau, \tau \to \tau\} \vdash x : \tau$
- Hence a term can have truly different (non-unifiable) types

**Very powerful at** charaterizing properties

***The* typing rule**

$$\frac{(\Gamma \vdash N : A_i)_{i \in 1..n} \quad A_i \neq A_j}{\Gamma \Vdash N : \{A_1, \ldots, A_n\}} \; e - multi$$

**Example**

**Let**

$$A = \tau \to \tau \qquad x : \{\boldsymbol{A \to A}, \boldsymbol{A}\} \vdash \boldsymbol{xx} : A \qquad \vdash \lambda x.x : \{\boldsymbol{A \to A}, \boldsymbol{A}\}$$

**Then**

$$\vdash (\lambda x.\boldsymbol{xx})(\lambda x.x) : A \qquad (\lambda x.\boldsymbol{xx})(\lambda x.x) \to_\beta \boldsymbol{(\lambda x.x)(\lambda x.x)}$$

# Idempotent Intersection Types a la Church

**Key idea**

- Variables can have multiple types **defined a priori** *e.g.* $x : \{\tau, \tau \to \tau\} \vdash x^{\tau} : \tau$
- Hence a term **modulo erasure** can have truly different (non-unifiable) types

**Motivation**

- $\lambda^{\mathtt{m}}$ is a la Church (easier syntactic analysis)
- abscence of standard correspondent Church version of Curry system

# Idempotent Intersection Types a la Church **Key changes**

## Type unicity

- $\Lambda^{\mathsf{e}}_{\cap}$ assigns **multiple** types to each term
- $\Lambda^{\mathsf{i}}_{\cap}$ assigns **one** type to each term

$$\frac{(\Gamma \vdash N : A_i)_{i \in 1..n} \quad A_i \neq A_j}{\Gamma \Vdash N : \{A_1, \dots, A_n\}}\ e \qquad \Longrightarrow \qquad \frac{(\Gamma \vdash s_i : A_i)_{i \in 1..n} \quad A_i \neq A_j}{\Gamma \Vdash \{s_1, \dots, s_n\} : \{A_1, \dots, A_n\}}\ i$$

## Reduction refinement

- $\Lambda^{\mathsf{e}}_{\cap}$ **agnostic** substitution
- $\Lambda^{\mathsf{i}}_{\cap}$ **depending** (on types) substitution

**Recall** $\Lambda^{\mathsf{e}}_{\cap}$ $\quad \vdash \lambda x.x : \{\boldsymbol{A} \to \boldsymbol{A}, \boldsymbol{A}\} \qquad (\lambda x.\boldsymbol{xx})(\lambda x.x) \to_\beta (\boldsymbol{\lambda x.x})(\boldsymbol{\lambda x.x})$

**Now** $\quad x : \{\boldsymbol{A} \to \boldsymbol{A}, \boldsymbol{A}\} \vdash x^{\boldsymbol{A \to A}} x^{\boldsymbol{A}} : A \qquad \vdash \lambda x^A.x : \boldsymbol{A \to A} \qquad \vdash \lambda x^\tau.x : \boldsymbol{A}$

**Then** $\quad (\lambda x^{\{\boldsymbol{A \to A}, \boldsymbol{A}\}}.x^{\boldsymbol{A \to A}} x^{\boldsymbol{A}})\{\lambda x^A.x, \lambda x^\tau.x\} \quad \to_\beta \quad (\lambda x^A.x)(\lambda x^\tau.x)$

**So** $\quad (\lambda x.t)s \to_\beta t[s/x] \quad \Longrightarrow \quad (\lambda x^{\vec{A}}.t)\vec{s} \to_\beta t\,[s_1/x^{A_1}] \dots [s_n/x^{A_n}]$

# Idempotent Intersection Types a la Church **Correspondence**

**Problem** Reducing the argument of an application

$\Lambda_\cap^{\mathtt{e}}$ no problem

$$ts \to_\beta ts'$$

$\Lambda_\cap^{\mathtt{i}}$

$$\begin{aligned} t\{s_1, s_2, \ldots, s_n\} &\to_\beta t\{s_1', s_2, \ldots, s_n\} \\ &\to_\beta t\{s_1', s_2', \ldots, s_n\} \\ &\to_\beta \ldots \\ &\to_\beta t\{s_1', s_2', \ldots, s_n'\} \end{aligned}$$

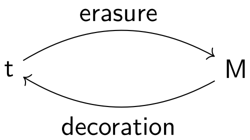
**Uniformity** $\vec{s}$ uniform if all $s_i$ are equal modulo erasure

**Refinement** $\vec{s}$ refines (noted $\sqsubset$) $t \in \Lambda_\cap^{\mathtt{e}}$ if uniform and $t = s_i$

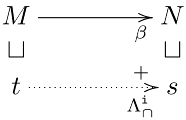
*e.g.* $\{\lambda x^\tau.x, \lambda x^A.x\} \sqsubset \lambda x.x$

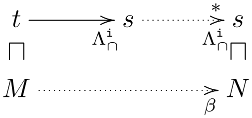
**Properties**

**Correspondence**




**Simulation**



# Introducing memories in $\Lambda^{\mathtt{i}}_{\cap}$

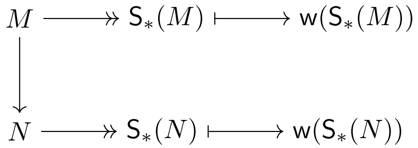
## Extension to $\lambda^{\mathtt{m}}_{\cap}$

- Addition of memories to the terms in $\Lambda^{\mathtt{i}}_{\cap}$
- Adaptation of definitions, properties and proofs of $\lambda^{\mathtt{m}}$ to multi-terms and multi-types

## Measure $\mathcal{W}_{\cap}$

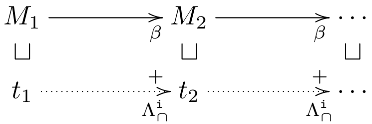
**Definition**

$$\mathcal{W}(M) = \mathsf{w}(\mathsf{S}_*(M))$$




## Strong Normalization of $\Lambda^{\mathtt{e}}_{\cap}$

- SN of $\Lambda^{\mathtt{i}}_{\cap}$
- Correspondence
- Simulation



# Conclusions and future work

**Conclusions**

- Overview of techniques for proving Strong Normalization
- Decreasing measures
- Auxiliar non-erasing $\lambda^m$ calculus, which allowed us to:
  - define $\mathcal{W}$: DM based on counting accumulated memories in $\lambda^{\mathtt{m}}$
  - extend $\mathcal{W}$ to $\Lambda_\cap$, obtaining a simpler measure than existing ones
  - generalize Turing's WN measure to SN by adding smaller measures of $D$-reachable terms

**Future work**

- Build a decreasing measure to System F
- Formalize them in a proof assistant
- Adapt $\mathcal{W}$ to idempotent intersection types characterizing head normal forms
- Further compare our measures with those by Gandy and de Vrijer

# Why "syntactic"

- sort of convention
- soft classification of SN proofs
- but...

| **semantic** | **syntactic** |
|---|---|
| reducibility (RC) | decreasing measures (DM)<br>reduction of SN to WN (NK) |

| **denotational** | **operational** |
|---|---|
| RC, de Vrijer | Gandy, NK |

| **denotational** | **operational** | **syntactic** |
|---|---|---|
| RC, DM | NK | RC, DM, NK |

- maybe **abstract** vs **concrete** would be better? **external** vs **internal**?
- we stick to the *soft* convention

  **syntactic** = "internal" analysis over the **structure of terms** or the **rewriting relation**

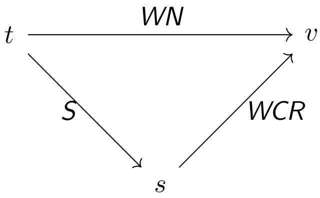
# The auxiliar $\lambda^m$−calculus

## Motivation

$\beta$ is erasing $\qquad (\lambda x.y)t \to_\beta y$

### A motivation not to erase

- Klop-Nederpelt lemma $\quad \mathit{INC} \land \mathit{WCR} \land \mathit{WN} \implies \mathit{SN} \land \mathit{CR}$
- We can obtain a decreasing measure from $\mathit{INC} \land \mathit{WCR} \land \mathit{WN}$
  - by WN there is a normal form $v$ for any $t$
  - by WCR it is the same for every reduct $s$ of $t$
  - by INC $\mathit{inc}(t) < \mathit{inc}(s) < \mathit{inc}(v)$
  - $\mathit{dec}(t) = \mathit{inc}(v) - \mathit{inc}(t)$



# Intuitive definition of $\mathcal{W}$

# Turing's measure "failing" example

## Example: copying a redex of greater degree

$$
\begin{aligned}
I_1 &= \lambda x^{\tau}.x & \delta(I_1\,x) &= \mathsf{h}(\tau \to \tau) &&= 1\\
I_2 &= \lambda x^{\tau\to\tau}.x & \delta(I_2\,I_1) &= \mathsf{h}((\tau \to \tau) \to (\tau \to \tau)) &&= 2\\
K &= \lambda x^{\tau}.\lambda y^{\tau}.x & \delta(K\,\_) &= \mathsf{h}(\tau \to \tau \to \tau) &&= 2\\
S_{K\,I} &= \lambda x^{\tau}.K\,x\,(I_1\,x) & \delta(S_{K\,I}\,\_) &= \mathsf{h}(\tau \to \tau) &&= 1
\end{aligned}
$$

$$
\mathcal{T}(\underbrace{S_{\underbrace{K}_{\mathsf{S2}}\,\underbrace{I}_{\mathsf{T1}}}\,(\underbrace{I_2\,I_1}_{\mathsf{U2}}\,x)}_{\mathsf{R1}}) = \{\underset{\mathsf{S}}{2}, \underset{\mathsf{U}}{2}, \underset{\mathsf{R}}{1}, \underset{\mathsf{T}}{1}\}
$$

$$
\mathcal{T}(\underbrace{K\,(\underbrace{I_2\,I_1}_{\mathsf{U'2}}\,x)}_{\mathsf{S2}}\,\underbrace{(I_1\,(\underbrace{I_2\,I_1}_{\mathsf{U'2}}\,x))}_{\mathsf{T1}}) = \{\underset{\mathsf{S}}{2}, \underset{\mathsf{U'}}{2}, \underset{\mathsf{U'}}{2}, \underset{\mathsf{T}}{1}\}
$$

# A first attempt: $\mathcal{T}'$ measure

**Problems**

$(>)$ A redex copies redexes of greater degree $\qquad \mathcal{T}(M) = [2,1] \longrightarrow \mathcal{T}(N) = [2,2]$

$(=)$ A redex copies redexes of same degree $\qquad \mathcal{T}(M) = [1,1] \longrightarrow \mathcal{T}(N) = [1,1]$

**Idea**

$i)$ generalize $\mathcal{T}$ to a family of measures $\mathcal{T}'_D$ indexed by a degree $D \in \mathbb{N}$, so *e.g.*

$$\mathcal{T}'_2(M) = [\underset{\mathsf{S}}{2}, \underset{\mathsf{R}}{1}] \qquad \text{and} \qquad \mathcal{T}'_1(M) = [\underset{\mathsf{R}}{1}]$$

$ii)$ instead of counting redex degrees in an isolated way,
consider also the information about remaining smaller redexes, so *e.g.*

$$\mathcal{T}'_2(M) = [\ (\underset{\mathsf{S}}{2}, \mathcal{T}'_1(M)),\ (\underset{\mathsf{R}}{1}, [])\ ] \qquad \mathcal{T}'_1(M) = [\ (\underset{\mathsf{R}}{1}, [])\ ]$$

## Definition

- $\mathcal{T}'_D(M) = [(i, \mathcal{T}'_{i-1}(M)) \mid R$ is a redex of degree $i \leq D$ in $M]$
- $\mathcal{T}'(M)\ = \mathcal{T}'_D(M)$ where $D$ is the maximum degree of $M$

# A first attempt: $\mathcal{T}'$ measure

*A working? example* $(>)$

**Definition**

- $\mathcal{T}'_D(M) = [(d, \mathcal{T}'_{d-1}(M)) \mid R$ is a redex of degree $d \leq D$ in $M]$
- $\mathcal{T}'(M) = \mathcal{T}'_D(M)$ where $D$ is the maximum degree of $M$

**Example**

$$M \quad = \quad \underbrace{S_{\underbrace{K}_{\text{S2}}\ \underbrace{I}_{\text{T1}}}\ (\underbrace{I_2\, I_1}_{\text{U2}}\, x)}_{\color{red}{\text{R1}}} \quad \longrightarrow_\beta \quad \underbrace{K\,(\underbrace{I_2\, I_1}_{\text{U}'2}\, x)}_{\text{S2}}\ \underbrace{(I_1\,(\underbrace{I_2\, I_1}_{\text{U}''2}\, x))}_{\text{T1}} \quad = \quad N$$

$$\mathcal{T}'_2(M) = [\ (\underset{\text{S}}{2}, \color{blue}{\mathcal{T}'_1(M)}),\ (\underset{\text{U}}{2}, \color{blue}{\mathcal{T}'_1(M)}),\ (\underset{\text{R}}{1}, []),\ (\underset{\text{T}}{1}, [])\ ] \qquad \mathcal{T}'_1(M) = \color{blue}{[\ (\underset{\text{R}}{1}, []),\ (\underset{\text{T}}{1}, [])\ ]}$$

$$\mathcal{T}'_2(N) = [\ (\underset{\text{S}}{2}, \color{blue}{\mathcal{T}'_1(M)}),\ (\underset{\text{U}'}{2}, \color{blue}{\mathcal{T}'_1(M)}),\ (\underset{\text{U}''}{2}, \color{blue}{\mathcal{T}'_1(M)}),\ (\underset{\text{T}}{1}, [])\ ] \qquad \mathcal{T}'_1(N) = \color{blue}{[\ (\underset{\text{T}}{1}, [])\ ]}$$

$$(2, \color{blue}{[\ (1, []),\ \ (1, [])\ ]}) \qquad > \qquad (2, \color{blue}{[\ (1, [])\ ]})$$

# A first attempt: $\mathcal{T}'$ measure

*A failing example* $(=)$

**Definition**

- $\mathcal{T}'_D(M) = [(d, \mathcal{T}'_{d-1}(M)) \mid R$ is a redex of degree $d \leq D$ in $M]$
- $\mathcal{T}'(M) = \mathcal{T}'_D(M)$ where $D$ is the maximum degree of $M$

**Example Example**

$$M \quad = \quad \underbrace{\underbrace{S_{\underbrace{K}_{\text{S2}}\,\underbrace{I}_{\text{T1}}}}\,\underbrace{(I_1\,x)}_{\text{U1}}}_{\text{R1}} \quad \longrightarrow_\beta \quad \underbrace{K\,\underbrace{(I_1\,x)}_{\text{U}'1}}_{\text{S2}}\;\underbrace{((\underbrace{I_1\,x}_{\text{U}''1}))}_{\text{T1}} \quad = \quad N$$

$$\mathcal{T}'_2(M) = [\ (\underset{\text{S}}{2}, \mathcal{T}'_1(M)),\ (\underset{\text{R}}{1}, []),\ (\underset{\text{T}}{1}, []),\ (\underset{\text{U}}{1}, []),\ ] \qquad \mathcal{T}'_1(M) = [\ (\underset{\text{R}}{1}, []),\ (\underset{\text{T}}{1}, []),\ (\underset{\text{U}}{1}, []),\ ]$$

$$\mathcal{T}'_2(N) = [\ (\underset{\text{S}}{2}, \mathcal{T}'_1(M)),\ (\underset{\text{T}}{1}, []),\ (\underset{\text{U}'}{1}, []),\ (\underset{\text{U}''}{1}, [])\ ] \qquad \mathcal{T}'_1(N) = [\ (\underset{\text{T}}{1}, []),\ (\underset{\text{U}'}{1}, []),\ (\underset{\text{U}''}{1}, [])\ ]$$

$$(2, [\ (1, []),\ (1, []),\ (1, [])\ ]) \qquad = \qquad (2, [\ (1, []),\ (1, []),\ (1, [])\ ])$$

# A second attempt: $\mathcal{T}^\beta$ measure

**Definition (development of a set of redexes)**

**reduction sequence** where each step corresponds to a **residual** of a redex **in the set**

- a **residual** is a copy of a redex left after contracting another
- notation: $\rho : m \xrightarrow{D}{}^{*}_{\beta} m'$

**Idea**

*i)* generalize $\mathcal{T}$ to a family of measures $\mathcal{T}^\beta_D$ indexed by a degree $D \in \mathbb{N}$

*ii)* instead of isolatedly counting redexes degrees, consider:
  - from set of redexes of degree $D$
  - target $M'$ from every development $\rho : M \xrightarrow{D}{}^{*}_{\beta} M'$
  - multiset of those $\mathcal{T}^\beta_{D-1}(M')$

**Definition**

$$\mathcal{T}^\beta_D(M) = [\ (i, \mathcal{V}^\beta_i(M)) \mid R \text{ is a redex of degree } i \leq D \text{ in } M\ ]$$

$$\mathcal{V}^\beta_D(M) = [\ \mathcal{T}^\beta_{D-1}(M') \mid \rho : M \xrightarrow{D}{}^{*}_{\beta} M'\ ]$$

**Problem**: our technique to prove it decreases does not work because of erasing

# A second attempt: $\mathcal{T}^\beta$ measure

**Definition**

$$\mathcal{T}^\beta_D(M) = [\ (i, \mathcal{V}^\beta_i(M)) \mid R \text{ is a redex of degree } i \leq D \text{ in } M\ ]$$

$$\mathcal{V}^\beta_D(M) = [\ \mathcal{T}^\beta_{D-1}(M') \mid \rho : M \xrightarrow{D}{}^{*}_{\beta} M'\ ]$$

**Reasoning about the auxiliar measure $\mathcal{V}^\beta_D$**

Consider

$$M \underset{R}{\to_\beta} N \qquad \mathcal{T}^\beta_D(M) > \mathcal{T}^\beta_D(N) \qquad \mathcal{V}^\beta_D(M) > \mathcal{V}^\beta_D(N)$$

1. Copying a redex of same degree $(=)$
   - injective mapping from devs of $\mathcal{V}^m_D(N)$ to devs of $\mathcal{V}^m_D(M)$ $\qquad R\rho : M \to_\beta N \to^*_\beta N'$

$$\mathcal{V}^\beta_D(M) > \mathcal{V}^\beta_D(N) \qquad \mathcal{T}^\beta_D(M) > \mathcal{T}^\beta_D(N)$$

2. Copying a redex of higher degree $(>)$
   - not clear the same can be done: a $\rho$ may erase $R$

$$\mathcal{V}^\beta_D(M') = \mathcal{V}^\beta_D(N') \qquad \mathcal{T}^\beta_D(M') = \mathcal{T}^\beta_D(N')$$

# $\mathcal{T}^m$ measure

**Idea**

$i)$ generalize $\mathcal{T}$ to a family of measures $\mathcal{T}^m_D$ indexed by a degree $D \in \mathbb{N}$

$ii)$ instead of isolatedly counting redexes degrees,
consider the multiset of the measures $\mathcal{T}^m_{D-1}$ of every target of a development of degree $D$

## Definition

$$\mathcal{T}^m_D(t) = [\ (i, \mathcal{V}^m_i(t)) \mid R \text{ is a redex of degree } i \leq D \text{ in } t\ ]$$

$$\mathcal{V}^m_D(t) = [\ \mathcal{T}^m_{D-1}(t') \mid \rho : t \xrightarrow{D}{}^{*}_{m} t'\ ]$$

**Lemmas**

- **Forget/decrease**: forgetful reduction $\triangleright$ decreases $\mathcal{T}^m$
- **High/increase**: contracting a redex of degree $D > i$ increases (non-strictly) $\mathcal{T}^m_i$
  only $\leq i$, no $D$, in $\mathcal{T}^m_i$ no erasing of any $\leq i$ maybe copies of $\leq i$
- **Low/decrease**: contracting a redex of degree $i < D$ decreases (strictly) $\mathcal{T}^m_D$
  injective mappings from devs of $\mathcal{V}^m_D(N)$ to devs of $\mathcal{V}^m_D(M)$

## Theorem

$$M \to_\beta N \quad \Longrightarrow \quad \mathcal{T}^m(M) > \mathcal{T}^m(N)$$